IMPULSIVE BOUNDARY VALUE PROBLEMS
and
IMPULSIVE CONTROL SYSTEMS

By

ALI SAEED S. AL-QAHTANI

April 2012

IMPULSIVE BOUNDARY VALUE PROBLEMS and IMPULSIVE CONTROL SYSTEMS

BY

ALI SAEED S. AL-QAHTANI

A Dissertation Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

April 2012

KING FAHD UNIVERSITY OF PETROLEUM & MINIRALS
DHAHRAN, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This dissertation, written by ALI SAEED SALIM AL-TALHAN AL-QAHTANI under the direction of his thesis advisors and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS.

Dissertation Committee

Prof. Abdelkader Boucherif
Dissertation Committee Chairman

Prof. Bilal Chanane
Co-Chairman

Prof. Nasser-eddine Tatar
Member

Prof. Mohamed El-Gebeily
Member

Dr. Khaled Furati
Member

Dr. Hattan Tawfiq
Department Chairman

Dr. Salam Zummo
Dean of Graduate Studies

13/5/12
Date

ACKNOWLEDGEMENTS

First and above all, I praise Allah, the almighty for providing me this opportunity and granting me the capability to proceed successfully.

Acknowledgement is due to King Fahd University of Petroleum & Minerals for supporting this research work and in particular, to Department of Mathematics and Statistics under the chairmanship of Dr. Hattan Tawfiq for giving me access to all available facilities which makes completion of this work much easier.

I wish to express my deep appreciation and gratitude to my advisors, Professor Abdelkader Boucherif and Professor Bilal Chanane, for introducing me to the subject and also for giving me all necessary support I needed to complete this work. Without their help this dissertation would not have been completed or written. My gratitude also goes to my committee members Professor Nasser-eddine Tatar, Professor Mohamed El-Gebeily and Dr. Khaled Furati for their valuable suggestions and constructive criticism.

I extend my thanks to my parents, my brothers, my sisters and my wife.

DEDICATION

I wish to dedicate this thesis to:

My parents, my brothers and my sisters for their care and continuous prayers for my success;

My wife for her care, patience and support to complete this work;

My lovely and intelligent daughters:

Kholod, Raghad, Layan, Lojain, Danah and Talah.

Contents

ABSTRACT (ENGLISH)

Name: Ali Saeed Salim Al-Talhan Al-Qahtani

Title of study: Impulsive Boundary Value Problems and Impulsive Control Systems

Major Field: Applied Mathematics

Advisor: Prof. Abdelkader Boucherif

Co-Advisor: Prof. Bilal Chanane

Date of degree: April, 2012

Ordinary and partial differential equations with impulsive effects describe phenomena in the applied sciences that admit sudden jumps in their states at some time instants called impulse moments. Most often the phenomena under consideration occur on finite time interval. This shows the importance of the study of two-point boundary value problems for impulsive differential equations. On the other hand solutions of impulsive differential system can experience instabilities. This has led to the introduction of controls to stabilize the system, and to the development of a whole area of investigation, called impulsive control systems.

In this work, we start by presenting a result of independent interest on integro-differential inequalities for functions with jump discontinuities, then we address the issue of existence and uniqueness of solutions of two-point boundary value problems for second order impulsive differential systems. We provide sufficient conditions on the nonlinearities that allow the use of topological methods to prove our main results. The second part is devoted to the study of impulsive control systems with controls as functions of bounded variation. Here, again, we introduce sufficient conditions on the controls that guarantee the stability of the impulsive control system. In the last part of the thesis we consider an initial-boundary value problem for an impulsive parabolic differential equation. The method of approach is based on Green's function and fixed

point theorems. Finally, we conclude our work by some concluding remarks and suggestions for future research.

ABSTRACT (ARABIC)

ملخص الرســـالة

الاســـــــــم : علي بن سعيد بن سالم آل طلحان القحطاني

عنوان الرســالة : مسائل القيم الحدودية الدفعية وأنظمة التحكم الدفعية

التخصـــــــص : رياضيات تطبيقية

تاريخ التخــرج : جمادى الآخرة ١٤٣٣ هـ — أبريل ٢٠١٢

المعادلات التفاضلية الدفعية العادية أو الجزئية تصف الأنظمة في العلوم التطبيقية والتي يحدث لها تغير مفاجئ في لحظات زمنية مختلفة تسمى لحظات الدفع. معظم هذه الأنظمة تظهر في فترات زمنية منتهية. هذا يبين أهمية دراسة مسائل القيم الحدودية الدفعية. من ناحية أخرى فإن حلول المعادلات التفاضلية الدفعية ليس بالضرورة مستقرة. هذا يقودنا إلى إيجاد دوال تحكم لجعل هذه الأنظمة مستقرة وهذا ما يسمى أنظمة التحكم الدفعية.

في هذه الرسالة قمنا في البداية بتقديم نتيجة مستقلة حول المتباينات التكاملية التفاضلية الخاصة بالدوال المنفصلة التي بها قفزات عند نقاط انفصالها. بعد ذلك قمنا بدراسة وجود ووحدانية الحل لأنظمة القيم الحدودية ذات النقطتين من الرتبة الثانية. النتائج الرئيسية أثبتناها بالاعتماد على عدد من الطرق التوبولوجية المتعلقة بمبدأ النقطة الثابتة. الجزء الثاني كان مخصص لدراسة أنظمة التحكم الدفعية التي فيها دالة التحكم محدودة التغير. وأيضا قدمنا الشروط الكافية على دالة التحكم حتى يكون النظام مستقر. في الجزء الأخير تطرقنا لمسائل القيم الابتدائية الحدودية الجزئية الدفعية. اعتمدنا على دالة جرين ونظريات النقطة الثابتة لإثبات النتائج الرئيسية في هذا الجزء. في ختام هذه الرسالة قدمنا عددا من الملاحظات والاقتراحات للأعمال والأبحاث المستقبلية.

Chapter 1

Introduction

Many phenomena in the applied sciences exhibit sudden behavior (impulsive changes) in their states at some time instants. We can mention mechanical systems with impact, heart beats, blood flows, population dynamics, industrial robotics, biotechnology, economics, etc. ...

These phenomena are described by ordinary or partial differential equations with impulsive effects. The time instants where the impulsive effects take place are called impulse moments. The study of boundary value problems for impulsive differential equations is very important since the evolution of the phenomenon occurs on a finite time interval. Very often, solutions to these impulsive differential systems may experience instabilities as time increases. This situation justifies the introduction of control parameters in the system being modeled. This has lead to the development of impulsive control problems.

It seems that the study of impulsive differential systems was initiated in the paper [87]. It is useful for the readers to introduce the definition of an impulsive differential system (see [72]).

There are three components:

* a continuous-time differential equation, which governs the state of the system between impulses,
* an impulse equation, which models the jump of the state of the system at the impulse moments,
* a jump function (or criterion).

Due to its importance in the applications the theory of impulsive differential systems has attracted the attention of many world class researchers. In fact, several monographs have been devoted to the study of impulsive differential systems with fixed impulse moments and state dependent impulse moments. We refer the interested reader to [16], [72], [95], [107]. There is an extensive list of research articles, mostly written by mathematicians, dealing with the subject. We mention, here, only those papers that are close to our interest, [1], [8], [40], [42], [45], [57], [58], [61], [64], [65], [71], [73], [79], [94], [96], [103], [118], [120], [128].

Almost all the above cited papers deal with the problem of existence of solutions of boundary value problems for second order impulsive differential equations.

On the other hand, the engineering community was interested in the study of control systems with impulse effects. The real development of impulsive control systems started during the late nineties. The interested reader can consult the monographs [85], [126] and the papers [4], [6], [23], [37], [41], [47], [52], [59], [66], [74], [76], [77], [80], [81], [86], [89], [100], [111], [132], [135].

It was pointed out in [126] that impulsive control systems can be classified into three types depending on the characteristic of the plant modeled by the differential system and the control laws.

The investigation partial differential equations with impulsive effects started with the pioneer work [46], devoted to the study of nonlinear parabolic partial differential equations with impulsive effects with application to growth of a population diffusing throughout its habitat and subject to abrupt changes such as harvesting, disasters and instantaneous stocking. After the publication of the paper [46] the mathematical literature has seen a rapid increase in the number of contributions to the theory of impulsive partial differential equations. We can mention the most recent contributions [11], [49], [60], [104], [114], [121], [134].

Our objective, in this dissertation, is to investigate the following problems:

A. Boundary value problems for second order impulsive control problems of the form

$$\begin{cases} x''(t) = F(t, x(t), x'(t)), \ t \in [0,T] \backslash \{t_1, t_2, ..., t_m\}, \\ \Delta x(t_k) = U_k(x(t_k)), \\ \Delta x'(t_k) = V_k(x'(t_k)), \ k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases}$$

where $x \in \mathbb{R}^n$ represents the state variable, F is a piecewise continuous function, U_k and V_k are the impulsive controls, and $0 < t_1 < t_2 < ... < t_m < T$. We shall provide sufficient conditions on the data F, U_k, V_k in order to obtain a priori bounds on solutions. We, then, rely on topological methods to prove existence and uniqueness of solutions.

B. Stability of impulsive control systems described by the following

$$\begin{cases} x'(t) = f(t, x, u) + g(t, x, u)u'(t), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where u is a control function of bounded variation and u' is the distributional derivative of u. We discuss the problem of existence, uniqueness and stability of solutions.

C. Impulsive problems for parabolic equations. Here, we consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t,u(x,t)), t \neq t_k, x \in [0,A], \\ u(x,0) = 0, \qquad x \in [0,A], \\ u(0,t) = u(A,t) = 0, \qquad t \in [0,T] \\ u(x,t_k^+) = u(x,t_k^-) + \sigma_k(x,t_k^-,u(x,t_k^-)), k = 1,2,...,m, \end{cases} \tag{1.2}$$

where f is a given function and σ_k represents the jump function at the point $(x, t_k), k = 1, 2, ..., m$. For the sake of simplicity, we will consider only the case $m = 1$.

The remainder of this dissertation is organized as follows. In chapter 2, we present a result, of independent interest, on inequalities for functions with jump discontinuity. Chapter 3 is devoted to the study of boundary value problems for second order impulsive control problems. Chapter 4 deals with the study of the practical stability of impulsive control systems with controls of bounded variation. The case of impulsive parabolic problems is investigated in chapter 5. Finally, we complete our work with some concluding remarks and suggestions for further research.

Chapter 2

Impulsive Integro-Differential Inequalities

2.1 Introduction

Integral inequalities play a fundamental role in the investigation of existence, uniqueness and stability properties of solutions of both ordinary and partial differential equations and initial value problems with impulse effects. For a good account and an excellent bibliography on the subject we refer the interested reader to [25], [26], [50], [83], [88] and [118].

In this chapter we present some results of independent interest a new integro-differential inequality for functions with jump discontinuities which will be applied to investigate the boundedness of solutions of an impulsive initial value problem for second order differential equations.

2.2 Preliminaries

In this section we introduce some definitions and notations that will be used in the remainder of the chapter. For more details see for instance [17].

Let t_0 and τ be nonnegative real numbers and let I denote the real interval $[t_0, \tau)$. For $i = 1, 2, ...$ consider the points $t_1, t_2, t_3, ..., t_m$ such that $0 \leq t_0 < t_1 < t_2 < ... < t_m < \tau$. If $J = \{t_i : i = 1, 2, ..., m\}$ let $I' = I \backslash J$. $PC(I)$ denotes the space of all functions $u : I \to \mathbb{R}$ continuous on I', and for $i = 1, 2, ..., m$ $u(t_i^+) = \lim\limits_{\epsilon \to 0+} u(t_i + \epsilon)$ and $u(t_i^-) = \lim\limits_{\epsilon \to 0} u(t_i - \epsilon)$ exist, and $u(t_i^-) = u(t_i)$. This is a Banach space when equipped with the sup-norm, i.e. $|u|_\infty = \sup\limits_{t \in I} |u(t)|$. Similarly, $PC^1(I)$ is the space of all functions $u \in PC(I)$, u is continuously differentiable on I', and for $i = 1, 2, ..., m$ $u'(t_i^+)$ and $u'(t_i^-)$ exist and $u'(t_i) = u'(t_i^-)$. For $u \in PC^1(I)$ we define its norm by $|u|_1 = \max\left(|u|_\infty, |u'|_\infty\right)$. Then $(PC^1(I), |\cdot|_1)$ is a Banach space.

2.3 Some Basic Inequalities

In this section we introduce some results about impulsive inequalities which are important.

Lemma 2.1 *[72] Let x and k be scalar nonnegative functions defined and integrable on $[a, b]$. Then, for any nonnegative constant c, the inequality*

$$x(t) \leq c + \int_a^t k(s)x(s)ds, \ t \in [a, b],$$

implies

$$x(t) \leq c \exp\left(\int_a^t k(s)ds\right), \ t \in [a, b].$$

2.3.1 Inequalities for Discontinuous Functions

Assume that

(A1) the sequence $\{t_k\}$ satisfies $t_0 < t_1 < t_2 < ...$, with $\lim_{k\to\infty} t_k = \infty$,

(A2) $x \in PC^1(\mathbb{R}^+, \mathbb{R})$.

Theorem 2.1 *[72]Assume that (A1) and (A2) hold. Suppose that q and $p \in C(\mathbb{R}^+, \mathbb{R})$ and for $k = 1, 2, ...,\ t \geq t_0$,*

$$\begin{cases} x'(t) \leq p(t)x(t) + q(t), & t \neq t_k, \\ x(t_k^+) \leq d_k x(t_k) + b_k, \end{cases} \tag{2.1}$$

where $d_k \geq 0$ and b_k are constants. Then

$$\begin{aligned} x(t) \;\leq\; & x(t_0)\left(\prod_{t_0<t_k<t} d_k\right)\exp\left(\int_{t_0}^{t} p(s)ds\right) \\ & + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k \\ & + \int_{t_0}^{t}\left(\prod_{s<t_k<t} d_k\right)\exp\left(\int_{s}^{t} p(\tau)d\tau\right) q(s)ds. \end{aligned} \tag{2.2}$$

Proof. Let $t \in [t_0, t_1]$. Then, we get from the first line of (2.1)

$$\frac{d}{dt}\left(x(t)\exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right)\right) \leq q(t)\exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right),$$

which yields after integrating from t_0 to t,

$$x(t) \leq x(t_0)\exp\left(\int_{t_0}^{t} p(\tau)d\tau\right) + \int_{t_0}^{t} q(s)\exp\left(\int_{s}^{t} p(\tau)d\tau\right)ds, \tag{2.3}$$

for $t_0 \leq t \leq t_1$. Hence (2.2) is true for $t \in [t_0, t_1]$. Now assume that (2.2) holds for $t \in [t_0, t_n]$ for some integer $n > 1$. Then, for $(t_n, t_{n+1}]$, it follows from the first inequality in (2.1) and (2.3) that

$$x(t) \leq x(t_n^+) \exp\left(\int_{t_n}^{t} p(\tau)d\tau\right) + \int_{t_n}^{t} q(s) \exp\left(\int_{s}^{t} p(\tau)d\tau\right) ds, \tag{2.4}$$

using the second inequality in (2.1), we obtain from (2.4)

$$x(t) \leq (d_n x(t_n) + b_n) \exp\left(\int_{t_n}^{t} p(\tau)d\tau\right) + \int_{t_n}^{t} q(s) \exp\left(\int_{s}^{t} p(\tau)d\tau\right) ds. \tag{2.5}$$

By the induction hypothesis (2.5) can be reduced to

$$\begin{aligned} x(t) \;\; \leq \;\; & d_n \exp\left(\int_{t_n}^{t} p(s)ds\right) \Bigg[x(t_0) \prod_{t_0<t_k<t_n} d_k \exp\left(\int_{t_0}^{t_n} p(s)ds\right) \\ & + \sum_{t_0<t_k<t_n} \left(\prod_{t_k<t_j<t_n} d_j \exp\left(\int_{t_k}^{t_n} p(s)ds\right)\right) b_k \\ & + \int_{t_0}^{t_m} \prod_{s<t_k<t_n} d_k \exp\left(\int_{s}^{t_n} p(\tau)d\tau\right) q(s)ds \Bigg] \\ & + b_n \exp\left(\int_{t_n}^{t} p(s)ds\right) + \int_{t_n}^{t} q(s) \exp\left(\int_{s}^{t} p(\tau)d\tau\right) ds, \end{aligned}$$

which on simplification gives (2.2) for $t \in [t_0, t_{n+1}]$. This completes the proof. ■

Theorem 2.2 *[88] Assume that (A1) and (A2) hold. Suppose that $p \in C(\mathbb{R}^+, \mathbb{R})$ and for $k = 1, 2, ...,$*

$$x(t) \leq c + \int_{t_0}^{t} p(s)x(s)ds + \sum_{t_0<t_k<t} \beta_k x(t_k), \;\; t \geq t_0, \tag{2.6}$$

where $\beta_k \geq 0$, and c are constants. Then

$$x(t) \leq c \prod_{t_0<t_k<t} (1 + \beta_k) \exp\left(\int_{t_0}^{t} p(s)ds\right), \;\; t \geq t_0. \tag{2.7}$$

Proof. Setting the right hand side of (2.6) equal to $v(t)$, we get

$$\begin{cases} v'(t) = p(t)x(t), t \neq t_k, \;\; v(t_0) = c \\ v(t_k^+) = v(t_k) + \beta_k x(t_k). \end{cases}$$

Since $x(t) \leq v(t)$, we then have

$$\begin{cases} v'(t) \leq p(t)v(t), t \neq t_k, \ v(t_0) = c \\ v(t_k^+) \leq (1+\beta_k)v(t_k). \end{cases}$$

By using Theorem (2.1) we get

$$x(t) \leq c \prod_{t_0<t_k<t} (1+\beta_k) \exp\left(\int_{t_0}^{t} p(s)ds\right), \ t \geq t_0.$$

This completes the proof. ■

In the following theorem we present another inequality for discontinuous functions.

Theorem 2.3 *Assume that (A1) and (A2) hold. Suppose that* $p \in C(\mathbb{R}^+, \mathbb{R}^+)$, $h \in PC(\mathbb{R}^+, \mathbb{R})$ *and*

$$x(t) \leq h(t) + \int_{t_0}^{t} p(s)x(s)ds + \sum_{t_0<t_k<t} \beta_k x(t_k), \ t \geq t_0,$$

where $\beta_k \geq 0$ *are constants. Then, for* $t \geq t_0$

$$\begin{aligned} x(t) \ \leq \ & h(t) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} (1+\beta_j) \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) \beta_k h(t_k) \\ & + \int_{t_0}^{t} \left(\prod_{s<t_k<t} (1+\beta_k)\right) \exp\left(\int_{s}^{t} p(\tau)d\tau\right) p(s)h(s)ds. \end{aligned}$$

Proof. See [72]. ■

2.4 A New Impulsive Integro-Differential Inequality

In this section we introduce a new inequality for discontinuous functions. Before that we introduce a class of bounding functions (see [28]).

Definition 2.1 *We say that w belongs to the class Ω if*

(i) *w is a positive continuous nondecreasing function on $[a, +\infty)$, for any $a \geq 0$,*

(ii) *for any constants $b \geq 0$ and $c > 0$*

$$\int_a^{+\infty} \frac{d\sigma}{b + cw(\sigma)} = \infty.$$

Remark 2.1

1- *Ω is nonempty since $w(\sigma) = \sigma$, $\forall \sigma \in [a, +\infty)$ is in Ω.*

2- *the function $\Psi : [a, +\infty) \to [0, +\infty)$ defined by*

$$\Psi(l) = \int_a^l \frac{d\sigma}{b + cw(\sigma)}$$

is strictly increasing, continuous and $\lim_{l \to \infty} \Psi(l) = +\infty$. Hence, Ψ^{-1} is well defined and strictly increasing on $[0, \infty)$.

Now, we introduce the main results of this chapter

Lemma 2.2 *Suppose that the following conditions hold*

1- $u(\cdot) \in C^2([t_0, \infty); \mathbb{R})$, $u(t_0) = u_0 \geq 0$ and $u'(t_0) = z_0 \geq 0$,

2- $w(\cdot) \in \Omega$,

3- $p(\cdot)$ is a positive continuous function,

4- the inequality $u''(t) \leq p(t)w(u(t))$ holds for all $t \geq t_0$.

Then, there is a constant M such that,

$$u(t) \leq M \quad \text{for all } t \in [t_0, \tau], \ \tau \geq t_0.$$

Proof. It is clear that

$$u(t) = u_0 + z_0(t - t_0) + \int_{t_0}^{t} (s - t_0)u''(s)ds, \;\; t \geq t_0. \tag{2.8}$$

Let $L = \tau - t_0$ and $a(t) = u_0 + z_0(t - t_0)$. Then (2.8) yields

$$u(t) \leq a(t) + L \int_{t_0}^{t} p(s)w(u(s))ds. \tag{2.9}$$

Set

$$v(t) = a(t) + L \int_{t_0}^{t} p(s)w(u(s))ds. \tag{2.10}$$

Then $u(t) \leq v(t)$, $v(t_0) = u_0$ and

$$v'(t) = z_0 + Lp(t)w(u(t)).$$

Since w is nondecreasing and p continuous on $[t_0, \tau]$, we get

$$v'(t) \leq z_0 + Kw(v(t)), \tag{2.11}$$

where $K = Lp_0$ and $p_0 = \sup_{t\in[t_0,\tau]} p(t)$. From (2.11) we deduce that

$$\frac{v'(t)}{z_0 + Kw(v(t))} \leq 1. \tag{2.12}$$

An integration from t_0 to t gives

$$\int_{t_0}^{t} \frac{v'(s)}{z_0 + Kw(v(s))} ds \leq \int_{t_0}^{t} ds = t - t_0 \leq L, \quad t \in [t_0, \tau], \tag{2.13}$$

or

$$\int_{z_0}^{v(t)} \frac{d\sigma}{z_0 + Kw(\sigma)} \leq L. \tag{2.14}$$

Hence $\Psi(v(t)) \leq L$ for all $t \in [t_0, \tau]$. Since $w \in \Omega$ it follows that

$$v(t) \leq \Psi^{-1}(L) := M \;\; \text{forall } t \in [t_0, \tau]. \tag{2.15}$$

Since $u(t) \leq v(t)$ for all $t \in [t_0, \tau]$, we obtain

$$u(t) \leq M \tag{2.16}$$

for all $t \in [t_0, \tau]$. ■

We, now, present the new inequality for functions with jump discontinuities.

Theorem 2.4 *Assume that the following conditions hold*

1- $p(\cdot)$ *is a positive continuous function,*

2- $w(\cdot) \in \Omega$,

3- $u(\cdot)$ *is a nonnegative function such that*

$$\begin{aligned} \textit{(i) } u(t_0) &= u_0 \geq 0, \\ \textit{(ii) } u'(t) &\leq c + \int_{t_0}^{t} p(s)w(u(s))ds + \sum_{t_\circ < t_i < t} \beta_i u(t_i) \end{aligned} \tag{2.17}$$

for all $t \in I$, $c \geq 0$, $\beta_i \geq 0$, $i = 1, 2, ..., m$.

Then there is a constant $M_i > 0$ *such that,*

$$u(t) \leq M_i \quad \textit{for all } t \in [t_i, t_{i+1}), \ i = 1, 2, ..., m.$$

Proof. Denote by $v(t)$ the right-hand side of (2.17), i.e.

$$v(t) = c + \int_{t_0}^{t} p(s)w(u(s))ds + \sum_{t_0 < t_i < t} \beta_i u(t_i). \tag{2.18}$$

For all $t \neq t_k$, $k = 1, 2, ..., m$, we have

$$u'(t) \leq v(t) \ , \quad v(t_0) = c,$$

and

$$v'(t) = p(t)w(u(t)).$$

Also for $t = t_k$, we have

$$v(t_k) = c + \int_{t_0}^{t_k} p(s)w(u(s))ds + \sum_{t_0<t_i<t_k} \beta_i u(t_i)$$

and

$$v(t_k^+) = c + \int_{t_0}^{t_k^+} p(s)w(u(s))ds + \sum_{t_0<t_i<t_k^+} \beta_i u(t_i).$$

So

$$v(t_k^+) - v(t_k) = \beta_k u(t_k),$$

$$v(t_k^+) = v(t_k) + \beta_k u(t_k).$$

First, we consider $t \in [t_0, t_1]$. We have that

$$v'(t) = p(t)w(u(t)).$$

Since $u'(t) \le v(t)$ it follows that

$$u(t) \le u_0 + \int_{t_0}^{t} v(s)ds$$

where $u(t_0) = u_0$. Let

$$z(t) = u_0 + \int_{t_0}^{t} v(s)ds,$$

so that

$$u(t) \le z(t) \quad \text{and } z'(t) = v(t).$$

Hence

$$v'(t) \le p(t)w(z(t)) \ \text{ for all } t \in [t_0, t_1].$$

Therefore

$$z''(t) \le p(t)w(z(t)), \quad \forall t \in [t_0, t_1].$$

Lemma 2.2 implies that there is a constant $M_0 > 0$ such that

$$u(t) \le M_0, \quad \forall t \in [t_0, t_1].$$

Next, consider $t \in (t_1, t_2]$. We have that

$$v'(t) = p(t)w(u(t)) \;\text{ and }\; v(t_1^+) = v(t_1) + \beta_1 u(t_1).$$

From (2.18) we see that

$$\begin{aligned} v(t_1) &= c + \int_{t_0}^{t_1} p(s)w(u(s))ds + \beta_1 u(t_1) \\ &\le c + \int_{t_0}^{t_1} p(s)w(M_0)ds := c_1. \end{aligned}$$

Then

$$v(t_1^+) \le c_1.$$

Since $u'(t) \le v(t)$ we have

$$\begin{aligned} u(t) &\le u(t_1) + \int_{t_1}^{t} v(s)ds \\ &\le M_0 + \int_{t_1}^{t} v(s)ds. \end{aligned}$$

Let

$$z(t) = M_0 + \int_{t_1}^{t} v(s)ds.$$

Then

$$u(t) \le z(t) \quad \text{ and } z'(t) = v(t), \;\forall t \in (t_1, t_2].$$

So

$$v'(t) \leq p(t)w(z(t)),\ \forall t \in (t_1, t_2],$$

and

$$z''(t) \leq p(t)w(z(t)), \quad \forall t \in (t_1, t_2].$$

Using Lemma 2.2 again there is a constant $M_1 > 0$ such that

$$u(t) \leq M_1, \quad \forall t \in (t_1, t_2].$$

Repeating this process we find that for all $t \in (t_i, t_{i+1}]$, $i = 1, 2, ..., m$, there exists a constant $M_i > 0$ such that

$$u(t) \leq M_i, \quad \forall t \in (t_i, t_{i+1}].$$

This completes the proof of Theorem 2.4. ■

We close this chapter by an application of our main result.

2.4.1 Applications

Consider the following second order initial value problem with impulsive effect,

$$\begin{cases} y''(t) = f(t, y(t)),\ t \neq t_k,\ k = 1, 2, ..., m, \\ y(t_k^+) = y(t_k) + I_k(y(t_k)), \\ y'(t_k^+) = y'(t_k) + J_k(y'(t_k)), \\ y(0) = y_0 \qquad , \qquad y'(0) = z_0. \end{cases} \tag{2.19}$$

Theorem 2.5 *Suppose the following conditions hold*

(H1) $f : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ *is continuous, there exist* $p : [0, \tau] \to \mathbb{R}$ *continuous and nonnegative and* $w \in \Omega$ *such that*

$$|f(t, y)| \leq p(t)w(|y|), \quad t \in [0, \tau].$$

(H2) $I_k : \mathbb{R} \to \mathbb{R}$ *is continuous and* $|I_k(y)| \leq m_0,\ k = 1, 2, ..., m.$

(H3) $J_k : \mathbb{R} \to \mathbb{R}$ *is continuous and there exists* $\beta_k > 0$ *such that*

$$|J_k(y'(t_k))| \leq \beta_k |y(t_k)|, \quad k = 1, 2, ..., m.$$

Then there exists $M > 0$ *such that any solution* y *of (2.19) satisfies* $|y(t)| \leq M$ *for all* $t \in [0, \tau]$.

Proof. It can be easily shown that any solution y of (2.19) satisfies

$$y'(t) = z_0 + \int_0^t f(s, y(s))ds + \sum_{0<t<t_k} J_k(y'(t_k)). \tag{2.20}$$

It follows from (2.20) that

$$|y'(t)| \leq |z_0| + \int_0^t |f(s, y(s))|\, ds + \sum_{0<t<t_k} |J_k(y'(t_k))|\,. \tag{2.21}$$

Notice that $|y(t)|' = y'(t)\frac{y(t)}{|y(t)|}$ for all $t \in G$, where $G := \{s \in [0, \tau];\ \ |y(s)| > 0\}\,.$

Thus

$$|y(t)|' \leq |y'(t)| \qquad \text{for all } t \in G. \tag{2.22}$$

Let $u_0 = |z_0|$ and let $u(t) = |y(t)|$ for all $t \in [0, \tau]$.

Then (2.21), (2.22) and conditions *(H1)*, *(H2)*, *(H3)* imply that

$$u'(t) \leq u_0 + \int_0^t p(s)w(u(s))ds + \sum_{0<t_k<t} \beta_k u(t_k), \quad t \in G. \tag{2.23}$$

In fact, 2.23 is valid for all $t \in [0, \tau]$, since for $t \in [0, \tau] \backslash G$ we have $y(t) = 0$, which implies that $|y(t)| = 0$ and $|y(t)|' = 0$.

Theorem 2.4 implies that there exists M_k, $k = 1, 2, ..., m$ such that

$$u(t) \leq M_k \qquad \text{for each } \ t \in (t_k, t_{k+1}].$$

Letting $M = \max\{M_k : k = 1, 2, ..., m\}$ we see that

$$|y(t)| = u(t) \leq M \qquad \text{for all } t \in [0, \tau].$$

This completes the proof of the theorem. ■

Chapter 3

Impulsive Control Problems

3.1 Introduction

An Impulsive control is a control based on impulsive differential equations. Impulsive control systems can be classified into three types depending on the control laws. The first type is impulsive control system given by

$$\begin{cases} x' = F(t,x), \quad t \neq t_k, \\ \Delta x = U_k(g(x)); \quad t = t_k, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function; $U_k(y) \in \mathbb{R}^n$ are control laws; $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous function, for every $k;\ k = 1,\ 2,\ ...;\ 0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...$; and $t_k \to \infty$ as $k \to \infty$.

The second type is impulsive control system given by

$$\begin{cases} x' = F(t,x,u), \quad t \neq t_k, \\ \Delta x = U_k(g(x,u)); \quad t = t_k, \end{cases} \tag{3.2}$$

where u is a control input. The third type is impulsive control given by

$$\begin{cases} x' = F(t, x, u), & t \neq t_k, \\ \Delta x = I_k(x); & t = t_k, \end{cases} \tag{3.3}$$

From these three systems, we can see that the control input in the first type takes place at the sudden change of some state variables, in the second type there are two kinds of control inputs: continuous control input which work on all the state variables and impulsive control input which works at the sudden changes; finally, in the third type the system is an impulsive system and the control is continuous and work on the differential equation (for more details see [126]).

In the following few lines we will point to some of the benefits of impulsive control. Impulsive control is importent because of [127]: "1) in some cases, the system cannot be controlled by using continuous control. For example, a government cannot change saving rates of its central bank every day. 2) In some cases, impulsive control is more efficient. For example, suppose that the population of a kind of bacterium and the density of a bactricide are two state variables of a system. We can control the population by instantaneously changing the density of the bactericide without enhance the drug resistance of bacteria which may be caused by continuous control." In [77], Liu, X. Z. said: " an essential benefit of impulsive control is that such controls may be simpler to implement and involve cheaper control mechanisms. For example, in rocket control, impulsive corrections of trajectories may involve mechanisms that are less complex than mechanisms that monitor and correct online the flight of the rocket. Thus if a mechanism for rocket trajectory control based on corrective impulses could be designed, such a mechanism would be less costly than continuous-time flight control mechanisms."

There is an extensive bibliography on the subject. We mention some of the most recent articles in the introduction of this thesis.

Our first objective, in this chapter, is to provide sufficient conditions on the data in order to insure the existence of at least one solution of the following problem

$$
\begin{cases}
(p(t)x'(t))' + q(t)x(t) = F(t, x(t), x'(t)), \quad t \neq t_k, t \in [0,1], \\
\Delta x(t_k) = U_k(x(t_k), x'(t_k)), \\
\Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \; k = 1, 2, ..., m, \\
x(0) = x(1) = 0,
\end{cases}
\tag{3.4}
$$

where $x \in \mathbb{R}$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a piecewise continuous function; U_k and V_k represent the jump discontinuities of x and x', respectively, at $t = t_k \in (0,1)$, called impulse moments, with $0 < t_1 < t_2 < ... < t_m < 1$. The second objective is to prove some existence results about the second order impulsive control problem of the form

$$
\begin{cases}
x''(t) = F(t, x(t), x'(t)), \quad t \neq t_k, t \in [0, T], \\
\Delta x(t_k) = U_k(x(t)), \\
\Delta x'(t_k) = V_k(x'(t)), \; k = 1, 2, ..., m, \\
x(0) = x(T) = 0,
\end{cases}
\tag{3.5}
$$

where $x \in \mathbb{R}^n$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function; U_k and V_k are impulsive controls and for every k; $k = 1, 2, ..., m; 0 < t_1 < t_2 < ... < t_k < ... < T$.

3.2 Preliminaries

In this section we introduce some definitions and notations that will be used in the remainder of the chapter.

Let J denote the real interval $[0,1]$. For $i = 1,2,...,m$, consider the points t_1, $t_2,...,t_m$ such that $0 < t_1 < t_2 < ... < t_m < 1$. If $I = \{t_i : i = 1,2,...,m\}$ let $J' = J - I$. $PC(J)$ denotes the space of all functions $x : J \to \mathbb{R}$ continuous on J', and for $i = 1,2,...,m$, $x(t_i^+) = \lim\limits_{\epsilon \to 0+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim\limits_{\epsilon \to 0} x(t_i - \epsilon)$ exist. We write $x(t_i^-) = x(t_i)$. This is a Banach space when equipped with the sup-norm, i.e. $\|x\|_0 = \sup\limits_{t \in J} |x(t)|$. Similarly, $PC^1(J)$ is the space of all functions $x \in PC(J)$, x is continuously differentiable on J', and for $i = 1,2,...,m$, $x'(t_i^+)$ and $x'(t_i^-)$ exist and $x'(t_i) = x'(t_i^-)$. For $x \in PC^1(J)$ we define its norm by $\|x\|_1 = \|x\|_0 + \|x'\|_0$. Then $(PC^1(I), \|\cdot\|_1)$ is a Banach space.

3.3 Second Order Impulsive Differential Equations

In this section we study the second order impulsive differential equation of the form (3.4).

3.3.1 Linear Equation

In this part we present results that will be useful in the rest of the section.

$$\begin{cases} (p(t)x'(t))' + q(t)x(t) = f(t), \quad t \neq t_k, t \in [0,1], \\ \Delta x(t_k) = U_k(x(t_k), x'(t_k)), \\ \Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \ k = 1,2,...,m, \\ x(0) = x(1) = 0, \end{cases} \tag{3.6}$$

In order to study (3.6) we first consider the problem without impulses

$$\begin{aligned} &(p(t)x'(t))' + q(t)x(t) = f(t), \ t \in [0,1] \\ &x(0) = x(1) = 0. \end{aligned} \tag{3.7}$$

We shall assume, throughout this section, that the following condition holds.

(H0) (i) $p \in C^1(J : \mathbb{R}), p(t) \geq p_0 > 0$, for all $t \in J$.

(ii) $q \in C(J : \mathbb{R}), q(t) \leq p_0\pi^2$, for all $t \in J$, and $q(t) < p_0\pi^2$ on a subset of J of positive measure.

Lemma 3.1 *If **(H0)** is satisfied, then for any nonzero $x \in C^2(J : \mathbb{R})$ with $x(0) = x(1) = 0$,*

$$\int_0^1 \{p(t)x'^2(t) - q(t)x^2(t)\}dt > 0.$$

Proof. The proof of this lemma is presented in [29]. We shall reproduce it here for the sake of completeness. Since $q(t) < p_0\pi^2$ on a subset of J of positive measure, we have that

$$p(t)x'^2(t) - q(t)x^2(t) > p_0(x'^2(t) - \pi^2x^2(t)).$$

This inequality yields

$$\int_0^1 \{p(t)x'^2(t) - q(t)x^2(t)\}dt > p_0 \int_0^1 \{x'^2(t) - \pi^2x^2(t)\}dt.$$

We show that

$$\mathcal{J}(x) = \int_0^1 \{x'^2(t) - \pi^2x^2(t)\}dt \geq 0$$

for all functions $x \in C^2(J : \mathbb{R})$ with $x(0) = x(1) = 0$. The function u that minimizes $\mathcal{J}(x)$ satisfies the Euler-Lagrange equation (see [44])

$$u'' + \pi^2u = 0,$$

and the boundary conditions $u(0) = u(1) = 0$. Then $u(t) = \sin \pi t$ or $u(t) = 0$, and $\mathcal{J}(u) = 0$. Since $\mathcal{J}(x) \geq \mathcal{J}(u)$ it follows that $\mathcal{J}(x) \geq 0$, and so

$$\int_0^1 \{p(t)x'^2(t) - q(t)x^2(t)\}dt > 0.$$

This completes the proof of the lemma. ∎

Lemma 3.2 *If **(H0)** is satisfied, the linear problem*

$$\begin{cases} (p(t)x'(t))' + q(t)x(t) = 0 \\ x(0) = x(1) = 0. \end{cases} \tag{3.8}$$

has only the trivial solution.

Proof. Assume on the contrary that problem (3.8) has a nontrivial solution x_0. Then (3.8) implies $[(p(t)x_0'(t))' + q(t)x_0(t)]\, x_0(t) = 0$ which yields

$$\begin{aligned} 0 &= \int_0^1 [(p(t)x_0'(t))' + q(t)x_0(t)]\, x_0(t)\ dt \\ &= \int_0^1 [(p(t)x_0'(t))']\, x_0(t)\ dt + \int_0^1 q(t)x_0^2(t)\ dt \\ &= -\int_0^1 \left[p(t)x_0'^2(t) - q(t)x_0^2(t)\right]\ dt < 0. \end{aligned}$$

This is a contradiction. See Lemma 3.1.

Therefore $x_0 \equiv 0$ is the only solution of (3.8). ■

It is well known that the unique solution of (3.7) is given by

$$x(t) = \int_0^1 G(t,s)f(s)ds,$$

where $G(\cdot,\cdot) : J \times J \to \mathbb{R}$ is the Green's function corresponding to the system (3.8).

Lemma 3.3 *The solution to (3.6) is*

$$\begin{aligned} x(t) \quad &= \quad \int_0^1 G(t,s)f(s)ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} p(t_k)U_k(x(t_k), x'(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)p(t_k)V_k(x(t_k), x'(t_k)), \quad \forall t \in [0,1]. \end{aligned} \tag{3.9}$$

Proof. We shall use the superposition principle and write $x(t) = y(t) + z(t) + w(t)$, where $y(t)$ solves the problem

$$\begin{cases} (p(t)y'(t))' + q(t)y(t) = f(t), \ t \in J, \\ \Delta y(t_k) = 0, \\ \Delta y'(t_k) = 0, \ k = 1, 2, ..., m, \\ y(0) = y(1) = 0, \end{cases} \tag{3.10}$$

$z(t)$ solves the problem

$$\begin{cases} (p(t)z'(t))' + q(t)z(t) = 0 \quad , \quad t \neq t_k, t \in J, \\ \Delta z(t_k) = U_k(x(t_k), x'(t_k)), \\ \Delta z'(t_k) = 0, \ k = 1, 2, ..., m, \\ z(0) = z(1) = 0, \end{cases} \tag{3.11}$$

and $w(t)$ solves the problem

$$\begin{cases} (p(t)w'(t))' + q(t)w(t) = 0 \quad , \quad t \neq t_k, t \in J, \\ \Delta w(t_k) = 0, \\ \Delta w'(t_k) = V_k(x(t_k), x'(t_k)), \ k = 1, 2, ..., m, \\ w(0) = w(1) = 0. \end{cases} \tag{3.12}$$

It is clear that

$$y(t) = \int_0^1 G(t,s)f(s)ds, \qquad t \in I.$$

For $k = 1, 2, ..., m$, set

$$z_k(t) = -\frac{\partial G(t, t_k)}{\partial s} p(t_k) U_k(x(t_k), x'(t_k)), \qquad t \in J,$$

and

$$w_k(t) = G(t, t_k) p(t_k) V_k(x(t_k), x'(t_k)), \qquad t \in J.$$

Using the properties of Green's function and its derivatives we can prove that the functions z_k and $w_k, k = 1, 2, ..., m$, are the solutions of problems (3.11) and (3.12), respectively.

Consequently, $x = y + \sum_{k=1}^{m} z_k + \sum_{k=1}^{m} w_k$ is a solution of problem (3.6). ■

3.3.2 Nonlinear Equation

In this section we present our main results on the existence of solutions for nonlinear boundary-value problems for second-order impulsive system.

Consider the problem

$$\begin{cases} (p(t)x'(t))' + q(t)x(t) = F(t, x(t), x'(t)), \quad t \neq t_k, t \in J, \\ \Delta x(t_k) = U_k(x(t_k), x'(t_k)), \\ \Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \ k = 1, 2, ..., m, \\ x(0) = x(1) = 0, \end{cases} \tag{3.13}$$

where $x \in \mathbb{R}$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a piecewise continuous function; U_k and V_k are impulsive functions representing the jump discontinuities of x and x' at $t \in \{t_1, t_2, ..., t_m\}$, respectively.

The nonlinear system

$$\begin{cases} (p(t)x'(t))' + q(t)x(t) = F(t, x(t), x'(t)) \\ x(0) = x(1) = 0, \end{cases} \tag{3.14}$$

is equivalent to the nonlinear integral equation

$$x(t) = \int_0^1 G(t, s)F(s, x(s), x'(s))ds, \text{ for all } t \in J$$

It follows from Lemma (3.3) that any solution of (3.13) satisfies

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m} W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)). \end{aligned} \tag{3.15}$$

where $W(t,t_k) = \frac{\partial G(t,t_k)}{\partial s}$. Let

$$\begin{aligned} K &= \max\{|G(t,s)| : (t,s) \in J \times J\}, \qquad L = \max\{|W(t,s)| : (t,s) \in J \times J\}, \\ M &= \sup\{\left|\frac{\partial G(t,s)}{\partial t}\right| : (t,s) \in J \times J\}, \quad N = \sup\{\left|\frac{\partial W(t,s)}{\partial t}\right| : (t,s) \in J \times J\}, \\ P &= \max\{K, L, M, N\}. \end{aligned}$$

For the next theorem we use the following assumptions:

(H1) $F(\cdot,\cdot,\cdot)$ is continuous on J' and satisfies the Lipschitz condition

$$|F(t,x_1,y_1) - F(t,x_2,y_2)| \leq \beta(|x_1 - x_2| + |y_1 - y_2|).$$

(H2) U_k and V_k are continuous and satisfy the Lipschitz conditions

$$\begin{aligned} |U_k(x_1,y_1) - U_k(x_2,y_2)| &\leq c_k\left(|x_1 - x_2| + |y_1 - y_2|\right), \\ |V_k(x_1,y_1) - V_k(x_2,y_2)| &\leq d_k\left(|x_1 - x_2| + |y_1 - y_2|\right), \end{aligned}$$

(H3) $2P\left(\beta + R\sum_{k=1}^{m} c_k + R\sum_{k=1}^{m} d_k\right) < 1$.

Theorem 3.1 *Under assumptions **(H0)-(H3)**, problem (3.13) has a unique solution.*

Proof. Define an operator $\omega : PC^1(J) \to PC^1(J)$ by

$$\begin{aligned}\omega(x)(t) &= \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m} W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)). \qquad (3.16)\end{aligned}$$

It is clear that any solution of (3.15) is a fixed point of ω and conversely any fixed point of ω is a solution of (3.15).

We shall show that ω is a contraction. Let $x, y \in PC^1(J)$, then

$$\begin{aligned}\|\omega(x)-\omega(y)\|_0 &\leq \sup_{t\in J}\{\int_0^1 |G(t,s)|\,|F(s,x(s),x'(s)) - F(s,y(s),y'(s))|\,ds \\ &\quad + \sum_{k=1}^{m} |W(t,t_k)|\,p(t_k)\,|U_k(x(t_k),x'(t_k)) - U_k(y(t_k),y'(t_k))| \\ &\quad + \sum_{k=1}^{m} |G(t,t_k)|\,p(t_k)\,|V_k(x(t_k),x'(t_k)) - V_k(y(t_k),y'(t_k))|\} \\ &\leq \sup_{t\in J}\{\int_0^1 |G(t,s)|\,(\beta\,(\|x-y\|_0 + \|x'-y'\|_0))ds \\ &\quad + R\sum_{k=1}^{m} |W(t,t_k)|\,c_k\,(\|x-y\|_0 + \|x'-y'\|_0) \\ &\quad + R\sum_{k=1}^{m} |G(t,t_k)|\,d_k\,(\|x-y\|_0 + \|x'-y'\|_0)\}.\end{aligned}$$

Now, by using **(H1)** and **(H2),** we have

$$\|\omega(x)-\omega(y)\|_0 \leq \beta K\,\|x-y\|_1 + RL\sum_{k=1}^{m} c_k\,\|x-y\|_1 + RK\sum_{k=1}^{m} d_k\,\|x-y\|_1\,. \quad (3.17)$$

We have

$$\begin{aligned}\frac{d}{dt}\omega(x)(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t}F(s,x(s),x'(s))ds - \sum_{k=1}^{m} \frac{\partial W(t,t_k)}{\partial t}U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial t}\;V_k(x(t_k),x'(t_k)).\end{aligned}$$

Let $x, y \in PC(J)$, then

$$\begin{aligned}\left\|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\right\|_0 &\leq \sup_{t\in J}\{\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| |F(s,x(s),x'(s)) - F(s,y(s),y'(s))|\, ds \\ &\quad + \sum_{k=1}^m \left|\frac{\partial W(t,t_k)}{\partial t}\right| |U_k(x(t_k),x'(t_k)) - U_k(y(t_k),y'(t_k))| \\ &\quad + \sum_{k=1}^m \left|\frac{\partial G(t,t_k)}{\partial t}\right| |V_k(x(t_k),x'(t_k)) - V_k(y(t_k),y'(t_k))|\}.\end{aligned}$$

Conditions **(H1)** and **(H2)** *imply*

$$\left\|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\right\|_0 \leq \beta M \|x-y\|_1 + RN\sum_{k=1}^m c_k \|x-y\|_1 + RM\sum_{k=1}^m d_k \|x-y\|_1 . \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\begin{aligned}\|\omega(x)-\omega(y)\|_1 &= \|\omega(x)-\omega(y)\|_0 + \left\|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\right\|_0 \\ &\leq \left(\beta K + RL\sum_{k=1}^m c_k + RK\sum_{k=1}^m d_k\right)\|x-y\|_1 \\ &\quad + \left(\beta M + RN\sum_{k=1}^m c_k + RM\sum_{k=1}^m d_k\right)\|x-y\|_1 \\ &\leq 2P\left(\beta + R\sum_{k=1}^m c_k + R\sum_{k=1}^m d_k\right)\|x-y\|_1\end{aligned}$$

Condition **(H3)** implies that ω is a contraction. By the Banach fixed point theorem ω has a unique fixed point x, which is the unique solution of (3.15). ■

For the next Theorem, we use the following assumptions:

(H4) $F : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous on $J'\times\mathbb{R}^2$ and there exists $h : J\times\mathbb{R}_+ \to \mathbb{R}_+$ a Caratheodory function, nondecreasing with respect to its second argument such that

$$|F(t,x,y)| \leq h(t,|x|+|y|), \quad \text{a.e. } t \in [0,1].$$

(H5) U_k and V_k are continuous and there exist $a_k > 0$ and $b_k > 0$ such that

$$|U_k(x(t_k),y(t_k))| \leq a_k \text{ and } \quad |V_k(x(t_k),y(t_k))| \leq b_k, \quad k = 1,2,...,m.$$

(H6) $\lim_{\varrho \to +\infty} \sup \frac{1}{\varrho}\left(\int_0^1 h(t,\varrho)dt + \sum_{k=1}^m R\left(a_k+b_k\right)\right) < \frac{1}{2P}.$

Theorem 3.2 *Under assumptions **(H0)**, **(H4)**-**(H6)**, problem (3.13) has at least one solution.*

Proof. The proof is given in two steps.

Step 1. A priori bound on solutions. Let $x \in PC^1(J)$ be a solution of (3.13).

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^m W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) \\ &+ \sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)), \ t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} x'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t}F(s,x(s),x'(s))ds - \sum_{k=1}^m \frac{\partial W(t,t_k)}{\partial t}p(t_k)U_k(x(t_k),x'(t_k)) \\ &+ \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial t}p(t_k)V_k(x(t_k),x'(t_k)), \ t \in [0,1],. \end{aligned}$$

It is easy to see that

$$\begin{aligned} |x(t)| &\leq K\int_0^1 |F(s,x(s),x'(s))|\, ds + RL\sum_{k=1}^m |U_k(x(t_k),x'(t_k))| \\ &+ RK\sum_{k=1}^m |V_k(x(t_k),x'(t_k))|\,, \end{aligned}$$

and

$$\begin{aligned} |x'(t)| &\leq M\int_0^1 |F(s,x(s),x'(s))|\,ds + RN\sum_{k=1}^m |U_k(x(t_k),x'(t_k))| \\ &\quad + RM\sum_{k=1}^m |V_k(x(t_k),x'(t_k))|\,. \end{aligned}$$

Conditions **(H4)**, ***(H5)*** *and* **(H6)** lead to

$$\|x\|_0+\|x'\|_0 \leq (K+M)\int_0^1 h(s,\|x\|_0+\|x'\|_0)ds+\sum_{k=1}^m R((L+N)l_k+(K+M)p_k).$$

Since $\|x\|_1 = \|x\|_0 + \|x'\|_0$ and h is nondecreasing, then

$$\|x\|_1 \leq 2P\int_0^1 h(s,\|x\|_1)ds+\sum_{k=1}^m R(2Pa_k+2Pb_k),$$

or

$$\|x\|_1 \leq 2P\left(\int_0^1 h(s,\|x\|_1)ds+\sum_{k=1}^m R\left(a_k+b_k\right)\right).$$

Let $\beta_0 = \|x\|_1$ .Then the above inequality gives

$$\frac{1}{2P} \leq \frac{1}{\beta_0}\left(\int_0^1 h(s,\beta_0)ds+\sum_{k=1}^m R\left(a_k+b_k\right)\right). \tag{3.19}$$

Condition **(H6)** implies that there exists $r > 0$ such that for all $\beta > r$ we have

$$\frac{1}{\beta}\left(\int_0^1 h(s,\beta)ds+\sum_{k=1}^m R\left(a_k+b_k\right)\right) < \frac{1}{2P}. \tag{3.20}$$

Comparing (3.19) and (3.20) we see that $\beta_0 \leq r$. Hence we have $\|x\|_1 \leq r$.

Step 2. Existence of solutions.

Let $\Omega = \{x \in PC^1(J) : \|x\|_1 < r+1\}$. Then Ω is an open convex subset of $PC^1(J)$. Define an operator H by

$$\begin{aligned} H(\lambda,x)(t) &= \lambda\int_0^1 G(t,s)F(s,x(s),x'(s))ds+\lambda\sum_{k=1}^m W(t,t_k)U_k(x(t_k),x'(t_k)) \\ &\quad +\lambda\sum_{k=1}^m G(t,t_k)V_k(x(t_k),x'(t_k)), \quad 0\leq\lambda\leq 1,\ t\in[0,1],. \end{aligned}$$

Then $H(\lambda,\cdot) : \bar{\Omega} \to PC^1(J)$ is compact and has no fixed point on $\partial\Omega$ (see [71]). It is an admissible homotopy between the constant map $H(0,\cdot) \equiv 0$ and $H(1,\cdot) \equiv \omega$. Since $H(0,\cdot)$ is essential then $H(1,\cdot)$ is essential which implies that $\omega \equiv H(1,\cdot)$ has a fixed point in Ω. This fixed point is a solution of our problem. ■

The following assumptions are used in the next theorem.

(H7) There exists $g \in L^1(J)$ such that

$$|F(t,x,y)| \leq g(t) \quad \text{for almost } t \in J,\ x, y \in \mathbb{R}.$$

(H8) $U_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $\alpha_k > 0$ such that

$$|U_k(x(t_k), y(t_k))| \leq \alpha_k \left(\|x\|_0 + \|y\|_0\right),\ k = 1, 2, ..., m.$$

(H9) $V_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $\beta_k > 0$ such that

$$|V_k(x(t_k), y(t_k))| \leq \beta_k \left(\|x\|_0 + \|y\|_0\right), \quad k = 1, 2, ..., m.$$

(H10) $2PR\sum_{k=1}^{m}(\alpha_k + \beta_k) < 1.$

Theorem 3.3 *Under assumptions **(H0)**, **(H7)-(H10)**, equation (3.13) has at least one solution.*

Proof. The proof is given in two steps.

Step1. A priori bound on solutions.

We have

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m} W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)),\ t \in [0,1],. \end{aligned}$$

and

$$x'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} F(s, x(s), x'(s))ds + \sum_{k=1}^{m} \frac{\partial W(t,t_k)}{\partial t} p(t_k) U_k(x(t_k), x'(t_k)) + \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial t} p(t_k) V_k(x(t_k), x'(t_k)),\ t \in [0,1],.$$

It is easy to see that

$$|x(t)| \le K \int_0^1 |F(s, x(s), x'(s))|\, ds + RL \sum_{k=1}^{m} |U_k(x(t_k), x'(t_k))| + RK \sum_{k=1}^{m} |V_k(x(t_k), x'(t_k))|,$$

and

$$|x'(t)| \le M \int_0^1 |F(s, x(s), x'(s))|\, ds + RN \sum_{k=1}^{m} |U_k(x(t_k), x'(t_k))| + RM \sum_{k=1}^{m} |V_k(x(t_k), x'(t_k))|.$$

From **(H7)**, **(H8)** and **(H9),** we obtain

$$\|x\|_0 + \|x'\|_0 \le (K+M)\|g\|_{L^1} + \sum_{k=1}^{m} R(L+N)\alpha_k(\|x\|_0 + \|x'\|_0) + \sum_{k=1}^{m} R(K+M)\beta_k(\|x\|_0 + \|x'\|_0).$$

Setting $\mu = 2PR\sum_{k=1}^{m}(\alpha_k + \beta_k)$, we obtain

$$\|x\|_1 \le 2P\|g\|_{L^1} + \mu\|x\|_1.$$

Then

$$(1-\mu)\|x\|_1 \le 2P\|g\|_{L^1}.$$

Using condition **(H10)** we obtain

$$\|x\|_1 \leq \left(\frac{2P}{1-\mu}\right)\|g\|_{L^1} := r_1.$$

Step 2. Existence of solutions.

Let $\Omega_1 = \{x \in PC^1(J) : \|x\|_1 < r_1 + 1\}$. The rest of the proof is similar to that of Theorem 3.2. ■

We end this chapter by some results about the second order impulsive control problems

3.4 Second Order Impulsive Control Problems

Control of impulsive differential equations appears naturally in physical phenomena. Most often these phenomena take place during a finite time interval. This leads to the study of boundary value problems for control of impulsive differential equations. In this section we address the problem of existence of solutions of control of impulsive differential equations of second order subjected to two-point boundary conditions. Our approach is based on the Granas topological transversality theorem and the Schauder fixed point theorem. The uniqueness of solutions is also discussed.

Let J denote the real interval $[0, T]$. For $i = 1, 2, ..., m$, consider the points t_1, $t_2, ..., t_m$ such that $0 < t_1 < t_2 < ... < t_m < T$. If $I = \{t_i : i = 1, 2, ..., m\}$ let $J' = J - I$. As we have seen before, $PC(J)$ denotes the space of all functions $x : J \to \mathbb{R}^n$ continuous on J', and for $i = 1, 2, ..., m$, $x(t_i^+) = \lim\limits_{\epsilon \to 0+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim\limits_{\epsilon \to 0} x(t_i - \epsilon)$ exist. We write $x(t_i^-) = x(t_i)$. This is a Banach space when equipped with the sup-norm, i.e. $\|x\|_0 = \sup\limits_{t \in J} | \, |x(t)| \, |$ where $| \, |x(\cdot)| \, |$ is the Euclidean norm of the vector $x(\cdot)$. Similarly, $PC^1(J)$ is the space of all functions $x \in PC(J)$, x is continuously differentiable on J', and for $i = 1, 2, ..., m$, $x'(t_i^+)$ and $x'(t_i^-)$ exist and

$x'(t_i) = x'(t_i^-)$. For $x \in PC^1(J)$ we define its norm by $\|x\|_1 = \|x\|_0 + \|x'\|_0$. Then $(PC^1(I), \|\cdot\|_1)$ is a Banach space.

In this section we consider the following second order impulsive control system

$$\begin{cases} x''(t) = F(t, x(t), x'(t)), \quad t \neq t_k, t \in [0, T], \\ \Delta x(t_k) = U_k(x(t)), k = 1, 2, ..., m, \\ \Delta x''(t_k) = V_k(x'(t)),\ k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases} \tag{3.21}$$

where $x \in \mathbb{R}^n$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function; U_k and V_k are impulsive controls and for every k; $k = 1, 2, ..., m; 0 < t_1 < t_2 < ... < t_k < ... < T$.

3.4.1 Linear Problem

This section is devoted to the study of the linear system corresponding to (3.21).

Consider the following linear second order impulsive control problem

$$\begin{cases} x'' = f(t), \quad t \neq t_k, t \in [0, T], \\ \Delta x(t_k) = U_k(x(t_k)), \\ \Delta x'(t_k) = V_k(x'(t_k)),\ k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases} \tag{3.22}$$

where $x \in \mathbb{R}^n$ is the state variable; U_k and V_k are impulsive controls and for every k; $k = 1, 2, ..., m; 0 < t_1 < t_2 < ... < t_k < ... < T$.

In order to study (3.22) we first consider the problem without impulses

$$\begin{cases} x'' = f(t),\ t \in [0, T], \\ x(0) = x(T) = 0 \end{cases} \tag{3.23}$$

It is well known that any solution of (3.23) is given by

$$x(t) = \int_0^T G(t,s)f(s)ds,$$

where $G(\cdot,\cdot) : [0,T]^2 \to \mathbb{R}$ takes the form

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T}, 0 \le s < t \le T; \\ \frac{t(s-T)}{T}, 0 \le t \le s \le T. \end{cases}$$

Lemma 3.4 *The solution of problem (3.22) is given by*

$$\begin{aligned} x(t) &= \int_0^T G(t,s)f(s)ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) \\ &+ \sum_{k=1}^{m} G(t,t_k)V_k(x'(t_k)). \end{aligned} \tag{3.24}$$

Proof. We shall use of supperposition principle and write $x(t) = y(t) + z(t) + w(t)$, where $y(t)$ solves the problem

$$\begin{cases} y'' = f(t), \quad t \in J', \\ \Delta y(t_k) = 0, \; k = 1,2,...,m, \\ \Delta y'(t_k) = 0, \; k = 1,2,...,m, \\ y(0) = y(T) = 0, \end{cases} \tag{3.25}$$

$z(t)$ solves the problem

$$\begin{cases} z'' = 0 \;\; , \;\; t \in J', \\ \Delta z(t_k) = U_k(x(t_k)), k = 1,2,...,m, \\ \Delta z'(t_k) = 0, \; k = 1,2,...,m, \\ z(0) = z(T) = 0, \end{cases} \tag{3.26}$$

and $w(t)$ solves the problem

$$\begin{cases} w'' = 0 \quad , \quad t \in J', \\ \Delta w(t_k) = 0, \;\; k = 1, 2, ..., m, \\ \Delta w'(t_k) = V_k(x'(t_k)), \;\; k = 1, 2, ..., m, \\ w(0) = w(T) = 0. \end{cases} \tag{3.27}$$

Then simple computations lead to (3.24). ■

Remark 3.1 *The Green's function $G(t, s)$ and its derivatives have the following properties (see [9] and [24])*

(i) $0 \le G(t, s) \le \frac{T}{4}$.

(ii) $\int_0^T G(t, s)ds \le \frac{T^2}{8}$.

(iii) $\int_0^T |G_t(t, s)|\, ds \le \frac{T}{2}$.

(iv) $|G_t(t, s)| \le 1$ *and* $|G_s(t, s)| \le 1$

(v) $\left|\frac{\partial^2 G(t,s)}{\partial s \partial t}\right| = \frac{1}{T}$.

3.4.2 Nonlinear Problem

In this section we will present our main results on the existence of solutions for nonlinear boundary value problems for second order of impulsive control system.

Consider the problem

$$\begin{cases} x'' = F(t, x, x'), \quad t \neq t_k, t \in [0, T], \\ \Delta x(t_k) = U_k(x(t_k)), \;\; k = 1, 2, ..., m, \\ \Delta x'(t_k) = V_k(x'(t_k)), \;\; k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases} \tag{3.28}$$

where $x \in \mathbb{R}^n$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function; U_k and V_k are impulsive controls with $0 < t_1 < t_2 < ... < t_m < T$.

It follows from Lemma (3.4) that any solution of (3.28) satisfies

$$\begin{aligned} x(t) &= \int_0^T G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)V_k(x'(t_k)),\ t \in [0,T]. \end{aligned} \tag{3.29}$$

Consider the following assumptions:

(H11) $F(\cdot,x,y)$ is continuous on J' and $F(t,\cdot,\cdot)$ satisfies a Lipschitz condition

$$\|F(t,x_1,y_1) - F(t,x_2,y_2)\| \le \alpha \|x_1 - x_2\| + \beta \|y_1 - y_2\|.$$

(H12) U_k, V_k are Lipschitz continuous with Lipschitz constant l_k and p_k, $k = 1,...,m$, respectively.

(H13) $\gamma T(T+4) < 8(1-\delta)$, where $\gamma = \max\{\alpha,\beta\}$ and $\delta = \max\{(1+\frac{1}{T})\sum_{k=1}^m l_k,$ $(\frac{T}{4}+1)\sum_{k=1}^m p_k\}$.

Theorem 3.4 *Under assumptions (H11)–(H13), problem (3.28) has a unique solution.*

Proof. Define an operator $\varphi : PC^1(J) \to PC^1(J)$ by

$$\begin{aligned} (\varphi x)(t) &= \int_0^T G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) \\ &\quad + \sum_{k=1}^{m} G(t,t_k)V_k(x'(t_k)),\ t \in [0,T]. \end{aligned}$$

It is clear that any solution of (3.29) is a fixed point of φ and vice-versa. We shall show that φ is a contraction. Let $x, y \in PC^1(J)$, then

$$\begin{aligned}
\|\varphi(x(t) - \varphi(y(t))\|_0 &\leq \int_0^T |G(t,s)| \, \|F(s,x(s),x'(s)) - F(s,y(s),y'(s))\|_0 \, ds \\
&\quad + \sum_{k=1}^{m} \left| \frac{\partial G(t,t_k)}{\partial s} \right| \|U_k(x(t_k)) - U_k(y(t_k))\|_0 \\
&\quad + \sum_{k=1}^{m} |G(t,t_k)| \, \|V_k(x'(t_k)) - V_k(y'(t_k))\|_0 \\
&\leq \int_0^T |G(t,s)| \, (\alpha \|x(s) - y(s)\|_0 + \beta \|x'(s) - y'(s)\|_0) ds \\
&\quad + \sum_{k=1}^{m} \left| \frac{\partial G(t,t_k)}{\partial s} \right| l_k \|x(t_k) - y(t_k)\|_0 \\
&\quad + \sum_{k=1}^{m} |G(t,t_k)| \, p_k \|x'(t_k) - y'(t_k)\|_0 .
\end{aligned}$$

Now, using conditions **(H11)**, **(H12)** and the above remark we get

$$\|\varphi(x) - \varphi(y)\|_0 \leq \gamma \frac{T^2}{8} \|x - y\|_1 + \sum_{k=1}^{m} l_k \|x - y\|_0 + \frac{T}{4} \sum_{k=1}^{m} p_k \|x' - y'\|_0 . \quad (3.30)$$

Next, we have that

$$\begin{aligned}
\frac{d}{dt}\varphi x(t) &= \int_0^T G_t(t,s) F(s,x(s),x'(s)) ds - \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial t \partial s} U_k(x(t_k)) \\
&\quad + \sum_{k=1}^{m} G_t(t,t_k) V_k(x'(t_k)),
\end{aligned}$$

The above inequality implies

$$\begin{aligned}
\left\|\frac{d}{dt}\varphi(x)-\frac{d}{dt}\varphi(y)\right\|_0 &\leq \sup_{t\in J}\{\int_0^T |G_t(t,s)|\,\|F(s,x,x')-F(s,y,y')\|_0\,ds \\
&\quad+\sum_{k=1}^{m}\left|\frac{\partial^2 G(t,t_k)}{\partial t\partial s}\right|\|U_k(x)-U_k(y)\|_0 \\
&\quad+\sum_{k=1}^{m}|G_t(t,t_k)|\,\|V_k(x')-V_k(y')\|_0\}.
\end{aligned}$$

Hence

$$\left\|\frac{d}{dt}\varphi x-\frac{d}{dt}\varphi y\right\|_0 \leq \gamma\frac{T}{2}\|x-y\|_1+\sum_{k=1}^{m}\frac{1}{T}l_k\|x-y\|_0+\sum_{k=1}^{m}p_k\|x'-y'\|_0. \quad (3.31)$$

From (3.30) and (3.31) we get

$$\begin{aligned}
\|\varphi x-\varphi y\|_1 &= \|\varphi(x)-\varphi(y)\|_0+\left\|\frac{d}{dt}\varphi(x)-\frac{d}{dt}\varphi(y)\right\|_0 \\
&\leq \gamma(\frac{T^2}{8}+\frac{T}{2})\|x-y\|_1+(1+\frac{1}{T})\sum_{k=1}^{m}l_k\|x-y\|_0 \\
&\quad+(\frac{T}{4}+1)\sum_{k=1}^{m}p_k\|x'-y'\|_0.
\end{aligned}$$

Letting $\delta=\max\{(1+\frac{1}{T})\sum_{k=1}^{m}l_k,\ (\frac{T}{4}+1)\sum_{k=1}^{m}p_k\}$, we get

$$\|\varphi x-\varphi y\|_1 \leq (\gamma(\frac{T^2}{8}+\frac{T}{2})+\delta)\|x-y\|_1.$$

It follows from condition **(H13)** that φ is contraction. By the Banach fixed point theorem φ has a unique fixed point x, which is the unique solution of (3.28). ■

Example 3.1 *Consider the impulsive control system*

$$\begin{cases} x''(t) = \frac{1}{2}\cos x(t), \;\; t \neq t_k, t \in J, \\ \Delta x(t_k) = \frac{5}{32k} x(t_k), \quad k = 1, 2, \\ \Delta x'(t_k) = \frac{k}{8} x'(t_k), \;\; k = 1, 2, \\ x(0) = x(1) = 0, \end{cases} \tag{3.32}$$

where $J = [0, 1]$, $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{4}$. *We see that* $\gamma = \frac{1}{2}$ *and* $\delta = \frac{15}{32}$. *So, for (3.32) we conclude that there is a unique solution. While, if we take* $\Delta x(t_k) = \frac{1}{k+1} x(t_k)$ *and* $\Delta x'(t_k) = \frac{1}{2k+1} x'(t_k)$. *Then we find* $\gamma = \frac{1}{2}$ *and* $\delta = \frac{5}{3}$. *We cannot conclude any thing about uniqueness since **(H13)** is not satisfied.*

For the next theorem we use the following assumptions:

(H14) $F(\cdot, x, y) : J \to \mathbb{R}^n$ is continuous on J', there exist $h : J \times \mathbb{R}_+ \to \mathbb{R}_+$ a Caratheodory function, nondecreasing with respect to its second argument such that

$$\|F(t, x(t), y(t))\| \le h(t, \|x\|_0 + \|y\|_0), \quad \text{a.e. } t \in [0, T].$$

(H15) $U_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $l_k > 0$ such that

$$\|U_k(x)\|_0 \le l_k, k = 1, 2, ..., m.$$

(H16) $V_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $p_k > 0$ such that

$$\|V_k(x)\|_0 \le p_k, \quad k = 1, 2, ..., m.$$

(H17) $\lim\limits_{\rho \to +\infty} \sup \frac{1}{\rho} \left(\int_0^T h(t, \rho) dt + \sum_{k=1}^m \left(\left(\frac{4(T+1)}{T(T+4)} \right) l_k + p_k \right) \right) < \frac{4}{T+4}$.

Theorem 3.5 *Under assumptions **(H14)**–**(H17)**, problem (3.28) has at least one solution.*

Proof. The proof is in two steps.

Step1. A priori bound on solutions.

$$\begin{aligned} x(t) &= \int_0^T G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial s} U_k(x(t_k)) \\ &+ \sum_{k=1}^m G(t,t_k)V_k(x'(t_k)),\ t \in [0,T]. \end{aligned}$$

and

$$\begin{aligned} x'(t) &= \int_0^T G_t(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial t \partial s} U_k(x(t_k)) \\ &+ \sum_{k=1}^m G_t(t,t_k)V_k(x'(t_k)),\ t \in [0,T]. \end{aligned}$$

It is easy to see that

$$\|x(t)\| \leq \frac{T}{4}\int_0^T \|F(s,x(s),x'(s))\| ds + \sum_{k=1}^m \|U_k(x(t_k))\| + \frac{T}{4}\sum_{k=1}^m \|V_k(x'(t_k))\|,$$

and

$$\|x'(t)\| \leq \int_0^T \|F(s,x(s),x'(s))\| ds + \frac{1}{T}\sum_{k=1}^m \|U_k(x(t_k))\| + \sum_{k=1}^m \|V_k(x'(t_k))\|.$$

Conditions **(H14),** ***(H15)*** *and* **(H16)** lead to

$$\|x\|_0 + \|x'\|_0 \leq (\frac{T}{4}+1)\int_0^T h(s,\|x\|_0 + \|x'\|_0)ds + \sum_{k=1}^m ((1+\frac{1}{T})l_k + (\frac{T}{4}+1)p_k).$$

Since $\|x\|_1 = \|x\|_0 + \|x'\|_0$ and h is nondecreasing, then

$$\|x\|_1 \leq (\frac{T}{4}+1)\int_0^T h(s,\|x\|_1)ds + \sum_{k=1}^m ((1+\frac{1}{T})l_k + (\frac{T}{4}+1)p_k).$$

Let

$$\rho_0 = \|x\|_1 .$$

Then the above inequality gives

$$\frac{4}{(T+4)} \leq \frac{1}{\rho_0}\left(\int_0^T h(s,\rho_0)ds + \sum_{k=1}^{m}\left(\left(\frac{4(T+1)}{T(T+4)}\right) l_k + p_k\right)\right). \tag{3.33}$$

Condition **(H17)** implies that there exists $r > 0$ such that for all $\rho > r$ we have

$$\frac{1}{\rho}\left(\int_0^T h(s,\rho)ds + \sum_{k=1}^{m}\left(\left(\frac{4(T+1)}{T(T+4)}\right) l_k + p_k\right)\right) < \frac{4}{(T+4)}. \tag{3.34}$$

Comparing (3.33) and (3.34) we can see that $\rho_0 \leq r$.

Hence, all possible solution of (3.28) satisfy

$$\|x\|_1 \leq r .$$

Step 2. Existence of solutions.

Let $\Omega = \{x \in PC^1(J) : \|x\|_1 < r+1\}$. Then Ω is an open convex subset of $PC^1(J)$. Define an operator $H : [0,1] \times \Omega \to PC^1(J)$ by

$$\begin{aligned} H(\lambda, x)(t) &= \lambda \int_0^T G(t,s)F(s,x(s),x'(s))ds - \lambda \sum_{k=1}^{m} \frac{\partial G(t,t_k)}{\partial t} U_k(x(t_k)) \\ &\quad + \lambda \sum_{k=1}^{m} G(t,t_k)V_k(x'(t_k)), \qquad 0 \leq \lambda \leq 1. \end{aligned}$$

$H(\lambda,\cdot) : \bar{\Omega} \to PC^1(J)$ is compact since it is the sum of two operators, the first one is a compact integral operator with kernel the Green's function,and the second is a finite rank operator, which is also compact (see [71], [94]). Also, it follows from the previous step that $H(\lambda,\cdot)$ has no fixed point on $\partial\Omega$, the boundary of Ω. Consequently, $H(\lambda,\cdot)$

is an admissible homotopy between the constant map $H(0,\cdot) \equiv 0$ and $H(1,\cdot) \equiv \varphi$. Since $H(0,\cdot)$ is essential then $H(1,\cdot)$ is essential, which implies that $\varphi \equiv H(1,\cdot)$ has a fixed point in Ω. This fixed point is a solution of our problem. ■

The following assumptions are used in the next theorem.

(H18) There exists $h \in L^1(J)$ such that

$$\|F(t,x(t),y(t))\| \leq h(t) \text{ for almost } t \in J.$$

(H19) $U_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and there exists $\alpha_k > 0$ such that

$$\|U_k(x)\|_0 \leq \alpha_k \|x\|_0, k = 1, 2, ..., m.$$

(H20) $V_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and there exists $\beta_k > 0$ such that

$$\|V_k(y)\|_0 \leq \beta_k \|y\|_0, \quad k = 1, 2, ..., m.$$

(H21) $\mu < 1$, where $\mu = \max\left\{\left(1+\frac{1}{T}\right)\sum_{k=1}^{m}\alpha_k, (\frac{T}{4}+1)\sum_{k=1}^{m}\beta_k\right\}$.

Theorem 3.6 *Under assumptions **(H18)**–**(H21)**, equation (3.28) has at least one solution.*

Proof. We proceed as in step 1 of the proof of the previous result to obtain a priori bound on solutions. It is clear that

$$\|x(t)\| \leq \frac{T}{4}\int_0^T \|F(s,x(s),x'(s))\|\, ds + \sum_{k=1}^{m}\|U_k(x(t_k))\| + \frac{T}{4}\sum_{k=1}^{m}\|V_k(x'(t_k))\|,$$

and

$$\|x'(t)\| \leq \int_0^T \|F(s,x(s),x'(s))\|\, ds + \frac{1}{T}\sum_{k=1}^{m}\|U_k(x(t_k))\| + \sum_{k=1}^{m}\|V_k(x'(t_k))\|.$$

(H18), **(H19)** and **(H20)** imply

$$\|x\|_0 + \|x'\|_0 \le (\frac{T}{4}+1)\|h\|_{L^1} + \sum_{k=1}^{m}\left(\left(1+\frac{1}{T}\right)\alpha_k\|x\|_0 + \left(\frac{T}{4}+1\right)\beta_k\|x'\|_0\right).$$

Letting $\mu = \max\left\{\left(1+\frac{1}{T}\right)\sum_{k=1}^{m}\alpha_k, (\frac{T}{4}+1)\sum_{k=1}^{m}\beta_k\right\}$ we get

$$\|x\|_1 \le (\frac{T}{4}+1)\|h\|_{L^1} + \mu\|x\|_1 .$$

Then

$$(1-\mu)\|x\|_1 \le (\frac{T+4}{4})\|h\|_{L^1} .$$

Condition **(H21)** gives

$$\|x\|_1 \le \frac{(T+4)}{4(1-\mu)}\|h\|_{L^1} .$$

Step 2. Existence of solutions.

Define a nonlinear operator $\psi : PC^1(J) \to X_0$ where $X_0 := \{x \in PC^1(J) : x(0) = x(T) = 0\}$ by

$$\begin{aligned}(\psi x)(t) &= \int_0^T G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^{m}\frac{\partial G(t,t_k)}{\partial s}U_k(x(t_k)) \\ &\quad + \sum_{k=1}^{m}G(t,t_k)V_k(x'(t_k)).\end{aligned}$$

Consider the closed convex set

$$D := \{x \in X_0 : \|x\|_1 \le \frac{(T+4)}{4(1-\mu)}\|h\|_{L^1}\}.$$

Then we can prove that ψ is continuous, maps D into itself and $\overline{T(D)}$ is compact. By the Schauder fixed point theorem, we conclude that ψ has a fixed point in D, which is a solution of our problem (3.28). ■

Chapter 4

Impulsive Control Problems with Controls of Bounded Variation

4.1 Introduction

4.1.1 Notation and Definitions

In this section we give some definetions and properties of functions of bounded variation and distributions. For more information and proofs of results see [68], [93], [106], [110], and [119].

Space of Functions of Bounded Variation

Let h be any function defined on an interval $I = [t_0, T]$ and consider all possible partitions of I as

$$S : t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T.$$

Definition 4.1 *The total variation of a function h on I is defined by*

$$V_{t_0}^T(h) = V(h, I) := \sup_S \left\{ \sum_{i=1}^m \|h(\alpha_i) - h(\alpha_{i-1})\| \right\},$$

where the supremum is taken over the set of all partitions of I and $\|\cdot\|$ is any norm on $\mathbb{R}^n$.

Definition 4.2 *The function h is called a function of bounded variation on I if the total variation of h on I is bounded, that is $V_{t_0}^T(h) < \infty$. Also h is called a function of bounded variation on $[t_0, \infty)$ if h has bounded variation on each interval $I_t = [t_0, t]$, $t_0 \leq t < \infty$.*

Definition 4.3 *The variation function of h on the interval I is defined by*

$$V_{t_0}^t(h) = V(h, I_t) := \sup_{S_t} \left\{ \sum_{i=1}^m \|h(\alpha_i) - h(\alpha_{i-1})\| \right\},$$

where $S_t : t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = t$ and $t < T$.

Now we present some important theorems about functions of bounded variation

Theorem 4.1 *[119]The set of discontinuity points of any function of bounded variation is a countable set.*

Theorem 4.2 *[106]Any function of bounded variation has a finite derivative almost everywhere.*

Theorem 4.3 *[119]Any right-continuous function of bounded variation $u(\cdot)$ on I generates the unique vector-valued measure defined on $(a, b] \subset I$ by the relation*

$$\mu((a, b]) = u(b) - u(a),$$

and

$$\mu(\{c\}) = u(c) - u(c^-), \quad \forall c \in (a, b].$$

Theorem 4.4 *[68]Let $F = \{u_\alpha(t)\}$ be some infinite functions defined on $[t_0, T]$. If all functions from F satisfy*

$$|u_\alpha(t)| \leq C, \quad V_{t_0}^T(u_\alpha(t)) \leq C, \quad C < \infty,$$

then there existes a sequence $u_{\alpha_n}(t) \in F$ which converges at each point $t \in [t_0, T]$ to some function $u(t)$ of bounded variation, such that

$$|u(t)| \leq C, \quad V_{t_0}^T(u(t)) \leq C.$$

Definition 4.4 *The space of all functions of bounded variation on I and taking values in $\mathbb{R}^n$ is defined as*

$$BV(I) = BV(I, \mathbb{R}^n) := \{h : I \to \mathbb{R}^n : V(h, I) < \infty\}.$$

Define the norm on $BV(I)$ as

$$\|h\|_{BV} = V(h, I) + \left\|h(t_0^+)\right\|.$$

$(BV(I), \|\cdot\|_{BV})$ is a Banach space [95].

Test Functions and Distributions

Let φ be a function defined on an open subset Ω of $\mathbb{R}^n$. The support of φ is the set $supp\varphi = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$. The set

$$D(\Omega) = \{\varphi \in C^\infty(\Omega) : supp\varphi \text{ is compact}\},$$

is a linear vector space, and is called the space of test functions. Any function belongs to $D(\Omega)$ is called test function.

Let f be a mapping defined on $D(\Omega)$ with value in $\mathbb{R}$ then f is called a functional. We write $\langle f, \varphi \rangle$ as the number assigned by f to $\varphi \in D(\Omega)$. A functional on D is said to be linear if

$$\langle f, \alpha\phi_1 + \beta\phi_2 \rangle = \alpha \langle f, \phi_1 \rangle + \beta \langle f, \phi_2 \rangle ,$$

for any two real numbers α, β and test functions ϕ_1, ϕ_2. The linear functional f is said to be continuous if for any sequence of test functions $\{\varphi_n\}_{n\in\mathbb{N}}$ that converges in D to zero, the sequence of numbers $\{\langle f, \varphi_n \rangle\}_{n\in\mathbb{N}}$ converges to zero.

Definition 4.5 *A continuous linear functional on $D(\Omega)$ is called a distribution. The space of all distributions is denoted by $D'(\Omega)$ and is called the dual space of D.*

Any distribution generated by a locally integrable function on $\mathbb{R}$, i.e. absolutely integrable on every finite interval in $\mathbb{R}$, is called regular distribution. Let f be a locally integrable function on $\mathbb{R}$ then f defines a distribution as follow

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)dt, \ \ \varphi \in D(\mathbb{R}).$$

Any distribution which is not regular is called singular.

Definition 4.6 *The distributional derivative of f is defined by*

$$\langle f', \phi \rangle = - \langle f, \phi' \rangle , \qquad \phi \in D(\Omega).$$

4.1.2 Review of Some Works Related to Our Study

Consider the system of the form

$$x'(t) = f(t, x(t)) + g(t, x(t))u'(t), \tag{4.1}$$

where $u \in BV$, the space of function of bounded variations, and denote by $u'(\cdot)$ the distributional derivative of u. Since u is a function of bounded variation, then u' can be considered as a Stieltjes measure. So it suddenly changes the state of the system at the points of discontinuity of u. Thus, we are dealing with an impulsive system.

In 1972, Das, P. C. and Sharma, R. R. [42] consider (4.1) as the perturbation of

$$x'(t) = f(t, x(t)), \tag{4.2}$$

They gave the proof of the equivalence between the equation (4.1) and the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} g(s, x(s))du(s). \tag{4.3}$$

After that, they gave a result about stability of (4.1) by using Lyapunov function method.

Leela, S. [75], in 1974, considered (4.1) as in [42]. The author studied the effect of impulsive perturbations on the stability and assumed that the set $x = 0$ is asymptotically self invariant (ASI) relative to (4.2) and gave sufficient conditions for the stability criteria of the ASI set $x = 0$ with respect to (4.1).

Pandit, S. G. [97], in 1977, considered (4.1) as in [42]. He used the fact that u can be written as $u_1 + u_2$ where u_1 is an absolutely continuous function of bounded variation and u_2 is a sum of jump functions, the jumps being those of u. In his work, he introduced a generalized Gronwall integral inequality and used it to prove stability of solutions of (4.1) with respect to the solution of (4.2).

After one year, that is in 1978, Rao, V. S. H. [103] consider the above system (4.1) as in [42] with $g(t, x) = p(t)$, where $p(t)$ is an integrable function with respect to u. He used second method of Lyapunov to obtain sufficient conditions for uniform boundedness of solutions of systems (4.1) and (4.2).

In [102], 1978, Rao, V. S. H., considered the problem (4.1) and the problem (4.2). He proved the asymptotically equivalence between (4.1) and (4.2) by assuming the existence of bounded solutions of these systems.

Pandit, S. G. [96], in 1980, considered system (4.1) with $g(t,x) = Ax + h(t,x)$ and he studied this system as the perturbation of $x'(t) = Ax(t)u'(t)$. He assumed that u has the form

$$u(t) = t + \sum_{k=1}^{\infty} a_k H_k(t); \;\; H_k(t) = \begin{cases} 0, & \text{for } t < t_k, \\ 1, & \text{for } t \geq t_k, \end{cases}$$

where $a_k \in \mathbb{R}$. In his work he developed a variation of parameters formula and he proved a result of asymptotic stability of system (4.1).

In 1987, Bressan A. in [35], considered (4.1) with f and g are C^1 and C^2 functions respectively. He proved that the map $\phi : u \to x_u$ can be continuously extended to bounded integrable control functions and the solution of (4.1) corresponding to a bounded integrable control function u is defined as the limit of the solutions corresponding to C^1-control functions u_n which converge to u in L^1.

In 1988, Bressan, A. and Rampazzo, F. [34] presented a definition of graph-completion of u denoted by φ. Then they defined the generalized solution, $x(\varphi, \cdot)$, of (4.1) relative to φ. Finaly they gave the relation between classical, when u is Lipschitz continuous function, and generalized solutions of (4.1).

Bressan, A. and Rampazzo, F. [33] in 1994, considered (4.1). In this work they studied this system when the vector fields g_i do not commute. They constructed a local factorization $A_1 \times A_2$ of the state space. Then they introduced a new commutative control system from (4.1) with single state x_1 and u, x_2 both playing the role of controls, where (x_1, x_2) are coordinates of $A_1 \times A_2$. They gave the relationship between this new system and system (4.1).

In [113], 1997, Shin, C. E., assumed that f in (4.1) is a bounded and Lipschitz continuous function and g is continuously differentiable with respect to x and satisfies $|g(t,\cdot)-g(s,\cdot)| \leq \varphi(t)-\varphi(s)$ for some increasing function φ and $s<t$. In his work, he defined a unique generalized solution of (4.1) corresponding to a measurable function of bounded variation under some assumptions on f and g.

In 2000, Shin, C. E. and Ryn, J. H. [111] considered (4.1) with

$$\sum_{i=1}^{m} |g_i(t_2,\cdot)-g_i(t_1,\cdot)| \leq \varphi(t_2)-\varphi(t_1)$$

for some increasing function φ and $t_1<t_2$. They defined a generalized solution $x_u(t)$ of (4.1) corresponding to u in the set $S_k := \{u=(u_1,\ldots,u_m):[0,T]\to\mathbb{R}^m$, each u_i is piecewise constant function such that u is right continuous, the discontinuities of u do not happen at the discontinuities of φ and the total variation of u is less than or equal to $k\}$. They proved that there exists $M>0$ such that $|x_u(T)-x_v(T)| \leq M\int\limits_0^T |u(s)-v(s)|\,d\varphi(s)$ for any $u,v\in S_k$ and for any $u,v\in C^1$.

In 2002, Filippova, T. F. in [47] , considered the problem

$$\begin{cases} x'(t)=f(t,x(t),u(t))+B(t,x(t))v'(t), & t\in[t_0,T], \\ x(t_0)=x_0, \end{cases} \tag{4.4}$$

where u is a measurable control function and v is an impulsive control function. In her work he studied the problem of the estimation of unknown states for a system of type (4.4) in an autonomus case with $f(t,x(t),u(t))=A(t)x$ and $B(t,x(t))=B(t)$ and she gave a result about the structure of the crossection of a trajectory tube.

We shall consider in this chapter a more general situation where the functions f and g depend also on u, thus

$$\begin{cases} x'(t)=f(t,x(t),u(t))+g(t,x(t),u(t))u'(t), & t\in[t_0,T], \\ x(t_0)=x_0, \end{cases} \tag{4.5}$$

where $u \in BV(I;\mathbb{R})$, $I = [t_0, T]$ and $f, g \in C(I \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$. In section 2 we give some preliminaries about functions of bounded variation. In section 3 we present some basic results that we need later in our work. Our main results about existence and stability of system (4.5) is presented in section 4.

4.2 Impulsive Control Problems with Controls of Bounded Variation

We consider a class of impulsive systems with controls as a functions of bounded variation. Using the topological transversality theorem and Banach contraction fixed point theorem, we derive existence results of solutions of the system. Stability of such systems is also investigated. Some examples are worked out to illustrate the approach.

4.2.1 Basic Results

In this section we summarize some basic results which we shall use later in this chapter.

Theorem 4.5 *Let $\{f_n\}$ be a sequence in $BV(I)$. Suppose there is a constant N such that*

$$\|f_n(t)\| \leq N \quad \text{for all } \; t \in I \; \text{and } n = 1, 2, ...,$$

and

$$V_{t_0}^{T}(f_n) \leq N \text{ for each } n.$$

Then there is a subsequence of $\{f_n\}$ which converges pointwise on I to a limit function f which is in $BV(I)$.

Proof. see [119]. ■

Remark 4.1 *We can see in Theorem 4.1 that if* $\|f_n(t)\| \leq N$ *for all* $t \in I$ *and* $n = 1, 2, ...,$ *then* $\|f_n(t_0^+)\| \leq N$ *and hence* $\|f_n\|_{BV} \leq 2N$.

Theorem 4.6 *[119] Suppose that* $f \in BV(I)$*. Then* f *is differentiable almost everywhere in* I*. Let us write* $\upsilon(t) = V_{t_0}^t(f)$*. If* E *is the subset of* (t_0, T) *on which* f *is differentiable, the function* $f' : E \to \mathbb{R}$ *is summable in the Lebesgue sense, and*

$$\int_E |f'| \, d\mu \leq \upsilon(T^-) - \upsilon(t_0^+),$$

where μ *is the Lebesgue measure.*

Definition 4.7 *By a solution of (4.1), say* $x(\cdot)$*, is meant a function of bounded variation whose distributional derivative* $x'(\cdot)$ *satisfies the equation (4.1).*

Lemma 4.1 *If* g *is an integrable function with respect to* μ*, and* F *is a distribution on* Ω *given by*

$$F(\varphi) = \int_\Omega \varphi d\mu, \quad \varphi \in D(\Omega),$$

then the product gF *defined by*

$$(gF)(\varphi) = \int_\Omega g\varphi d\mu, \quad \varphi \in D(\Omega),$$

is also a distribution on Ω.

Proof. See [42]. ■

Lemma 4.2 *A function* $x(\cdot) = x(\cdot, t_0, x_0)$ *is a solution of (4.5) if and only if it satisfies the integral equation*

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t g(s, x(s), u(s))du(s). \tag{4.6}$$

Proof. See [42] or [95]. ■

We introduce the following assumptions.

(H1) $f : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous in (t, x) for all u and satisfies

$$\|f(t, x, u)\| \leq |u| \, \|x\| \, \alpha(t)$$

where $\alpha(\cdot) > 0$ is a continuous function on $[t_0, T]$, and

$$\|f(t, x, u) - f(t, y, u)\| \leq R_1 \, |u| \, \|x - y\| \, .$$

(H2) $g : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous in (t, x) for all u and satisfies

$$\|g(t, x, u)\| \leq K,$$

and

$$\|g(t, x, u) - g(t, y, u)\| \leq R_2 \, \|x - y\| \, .$$

for some constant $R_2 > 0$.

Lemma 4.3 *Assume that **(H1)** and **(H2)** hold. Then any solution of system (4.5) is bounded.*

Proof. Let $x(\cdot)$ be a solution of (4.5). By Lemma 4.2 we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds + \int_{t_0}^{t} g(s, x(s), u(s))du(s).$$

Then

$$\begin{aligned}
\|x(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|f(s, x(s), u(s))\| \, ds + \int_{t_0}^{t} \|g(s, x(s), u(s))\| \, d \left| V_{t_0}^{s} u \right| \\
&\leq \|x_0\| + \int_{t_0}^{t} |u(s)| \, \|x(s)\| \, \alpha(s) ds + K \int_{t_0}^{t} d \left| V_{t_0}^{s} u \right| \\
&\leq \|x_0\| + K \left| V_{t_0}^{T} u \right| + \int_{t_0}^{t} |u(s)| \, \|x(s)\| \, \alpha(s) ds.
\end{aligned}$$

By using Gronwall's inequality, we get

$$\|x(t)\| \leq \left(\|x_0\| + K \left|V_{t_0}^T u\right|\right) e^{\bar{\alpha}\int_{t_0}^T |u(s)|ds},$$

where $\bar{\alpha} = \sup\{\alpha(t); t \in I\}$. Since u is a function of bounded variation then it is bounded, i.e. there existe $M > 0$ such that $|u(s)| \leq M$ for all $s \in [t_0, T]$. The last inequality leads to

$$\|x(t)\| \leq \left(\|x_0\| + K \left|V_{t_0}^T u\right|\right) e^{\bar{\alpha} M A},$$

where $A = T - t_0$. This shows that $x(t)$ is bounded. ■

Lemma 4.4 *Assume **(H1)** and **(H2)** hold. Then any solution of system (4.5) is a function of bounded variation.*

Proof. Consider any partition of the interval $I : t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T$. Then

$$\begin{aligned} \|x(\alpha_i) - x(\alpha_{i-1})\| &\leq \int_{\alpha_{i-1}}^{\alpha_i} \|f(s, x(s), u(s))\| \, ds + \int_{\alpha_{i-1}}^{\alpha_i} \|g(s, x(s), u(s))\| \, d\left|V_{t_0}^s u\right| \\ &\leq \int_{\alpha_{i-1}}^{\alpha_i} |u(s)| \, \|x(s)\| \, \alpha(s) ds + K \int_{\alpha_{i-1}}^{\alpha_i} d\left|V_{t_0}^s u\right| \end{aligned} \tag{4.7}$$

By taking summation from $i = 1$ to m, for both sides of the above inequality, we get

$$\begin{aligned} \sum_{i=1}^m \|x(\alpha_i) - x(\alpha_{i-1})\| &\leq \sum_{i=1}^m \int_{\alpha_{i-1}}^{\alpha_i} |u(s)| \, \|x(s)\| \, \alpha(s) ds + K \sum_{i=1}^m \int_{\alpha_{i-1}}^{\alpha_i} d\left|V_{t_0}^s u\right| \\ &\leq \int_{t_0}^T |u(s)| \, \|x(s)\| \, \alpha(s) ds + K \left|V_{t_0}^T u\right| \\ &\leq \bar{x} \; \bar{\alpha}(V_{t_0}^T |u|) + K \left|V_{t_0}^T u\right|. \end{aligned} \tag{4.8}$$

Hence

$$\sum_{i=1}^m \|x(\alpha_i) - x(\alpha_{i-1})\| \leq (\bar{x} \; \bar{\alpha} + K) V_{t_0}^T |u|,$$

where $\bar{x} = \sup\{x(t); t \in I\}$. In last inequality (4.8) the right hand side is finite because the variation of $|u|$ on I is bounded. Thus by taking supremum of both sides over any partition of I we get $V_{t_0}^T x < \infty$, that is $x \in BV(I)$. ■

Define the operator

$$\varphi(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds + \int_{t_0}^{t} g(s, x(s), u(s))du(s). \tag{4.9}$$

Lemma 4.5 *Assume that **(H1)** and **(H2)** hold. If $x \in BV(I)$ then $\varphi(x) : [t_0, T] \to \mathbb{R}^n$ given by (4.9) is a function of bounded variation on I.*

Proof. Assume that $\alpha_i, \alpha_j \in [t_0, T]$ such that $\alpha_i < \alpha_j$, then

$$\begin{aligned}
\|\varphi(x)(\alpha_i) - \varphi(x)(\alpha_j)\| &= \left\| \int_{\alpha_i}^{\alpha_j} f(s, x(s), u(s))ds + \int_{\alpha_i}^{\alpha_j} g(s, x(s), u(s))du(s) \right\| \\
&\leq \int_{\alpha_i}^{\alpha_j} \|f(s, x(s), u(s))\| \, ds + \int_{\alpha_i}^{\alpha_j} \|g(s, x(s), u(s))\| \, dV_{t_0}^s u \\
&\leq \int_{\alpha_i}^{\alpha_j} |u(s)| \, \|x(s)\| \, \alpha(s)ds + K \int_{\alpha_i}^{\alpha_j} dV_{t_0}^s u.
\end{aligned}$$

Now, for any partition of I, $t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T$, we have

$$\sum_{i=1}^{m} \|\varphi(x)(\alpha_i) - \varphi(x)(\alpha_{i-1})\| \leq \int_{t_0}^{T} |u(s)| \, \|x(s)\| \, \alpha(s)ds + KV_{t_0}^T u. \tag{4.10}$$

Since $\alpha(\cdot)$ is continuous on I and $u \in BV(I)$, then the right hand side of (4.10) is finite, and we conclude that $V_{t_0}^T \varphi(x) < \infty$, that is $\varphi(x) \in BV(I)$. ■

Theorem 4.7 *Assume that **(H1)** and **(H2)** hold. The mapping $\varphi : BV(I) \to BV(I)$, defined by (4.9), is a bounded compact operator on $BV(I)$.*

Proof. First, we prove that φ is bounded on $BV(I)$, i.e. there exists a constant $\beta > 0$ such that $\|\varphi\|_{BV} \leq \beta$ for all $\varphi \in BV(I)$. By the definition of $\|\cdot\|_{BV}$ we have

$$\|\varphi\|_{BV} = V_{t_0}^{T}\varphi + \left\|\varphi(x(t_0^+))\right\|.$$

Now

$$\begin{aligned}
\left\|\varphi(x(t_0^+))\right\| &= \left\| x_0 + \int_{t_0}^{t_0^+} f(s,x(s),u(s))ds + \int_{t_0}^{t_0^+} g(s,x(s),u(s))du(s) \right\| \\
&\leq \|x_0\| + \int_{t_0}^{t_0^+} |u(s)| \, \|x\| \, \alpha(s)ds + K \int_{t_0}^{t_0^+} dV_{t_0}^{s} u \\
&\leq \|x_0\| + KV_{t_0}^{T} u. \qquad (4.11)
\end{aligned}$$

From (4.10) and (4.11) we get that

$$\|\varphi\|_{BV} \leq \|x_0\| + L + KV_{t_0}^{T} u,$$

where $L = \int_{t_0}^{T} |u(s)| \, \|x(s)\| \, \alpha(s)ds$. Put $\beta = \|x_0\| + L + KV_{t_0}^{T} u$, then

$$\|\varphi\|_{BV} \leq \beta < \infty.$$

Therefore the operator $\varphi : BV(I) \to BV(I)$ is a bounded operator on the space $BV(I)$.

Next, we prove that φ is compact. Assume that $x_k \in BV(I), k = 1, 2, ...,$ is a bounded sequence in $BV(I)$, i.e. there exists a constant c such that $\|x_k\|_{BV} \leq c, k = 1, 2, ...$ By Theorem 4.5, the sequence $\{x_k\}$ contains a subsequence $\{x_{k_l}\}$ which converges pointwise to a function $\tilde{x} \in BV(I)$, i.e.

$$\lim_{k\to\infty} x_{k_l}(t) = \tilde{x}(t), \ \forall t \in I. \qquad (4.12)$$

Define

$$y(t) = x_0 + \int_{t_0}^{t} f(s,\tilde{x}(s),u(s))ds + \int_{t_0}^{t} g(s,\tilde{x}(s),u(s))du(s), \qquad (4.13)$$

so by Lemma 4.2, $y \in BV(I)$. Put $z_l(t) = x_{k_l}(t) - \tilde{x}(t)$, $t \in I$. Then $z_l(t) \in BV(I)$ and by (4.12) we have

$$\lim_{l\to\infty} z_l(t) = 0, \ \forall t \in I. \tag{4.14}$$

Also,

$$\begin{aligned}
\|z_l(t)\| &\leq \left\|z_l(t) - z_l(t_0^+)\right\| + \left\|z_l(t_0^+)\right\| \\
&\leq V_{t_0}^T z_l + \left\|z_l(t_0^+)\right\| \\
&\leq \|z_l\|_{BV} \\
&\leq \|x_{k_l} - \tilde{x}\|_{BV} \\
&\leq c + \|\tilde{x}\|_{BV}\ , \qquad \forall t \in I.
\end{aligned}$$

Let $|u| \leq M$. Using definition of φ and y we get

$$\begin{aligned}
\left\|\varphi(x_{k_l}(t_0^+)) - y(t_0^+)\right\| &\leq R_1 M \int_{t_0}^{t_0^+} \|x_{k_l} - \tilde{x}\|\, ds + R_2 \int_{t_0}^{t_0^+} \|x_{k_l} - \tilde{x}\|\, dV_{t_0}^s u \\
&\leq R_1 M \int_{t_0}^{t_0^+} \|z_l\|\, ds + R_2 \int_{t_0}^{t_0^+} \|z_l\|\, dV_{t_0}^s u.
\end{aligned}$$

Taking the limit of both sides of the inequality we obtain

$$\lim_{l\to\infty} \left\|\varphi(x_{k_l}(t_0^+)) - y(t_0^+)\right\| = 0. \tag{4.15}$$

Now, let $\alpha_i, \alpha_j \in [t_0, T]$ such that $\alpha_i < \alpha_j$, then

$$\begin{aligned}
&\|[\varphi(x_{k_l}(\alpha_j)) - y(\alpha_j)] - [\varphi(x_{k_l}(\alpha_i)) - y(\alpha_i)]\| \\
&\qquad \leq \int_{\alpha_i}^{\alpha_j} \|f(s, x_{k_l}(s), u(s)) - f(s, \tilde{x}(s), u(s))\|\, ds \\
&\qquad\quad + \int_{\alpha_i}^{\alpha_j} \|g(s, x_{k_l}(s), u(s)) - g(s, \tilde{x}(s), u(s))\|\, du(s) \\
&\qquad \leq R_1 M \int_{\alpha_i}^{\alpha_j} \|z_l\|\, ds + R_2 \int_{\alpha_i}^{\alpha_j} \|z_l\|\, dV_{t_0}^s u.
\end{aligned}$$

For any partition of I as $t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T$, we get

$$\sum_{i=1}^{m} \|[\varphi(x_{k_l}(\alpha_i)) - y(\alpha_i)] - [\varphi(x_{k_l}(\alpha_{i-1})) - y(\alpha_{i-1})]\|$$

$$\leq R_1 M \int_{t_0}^{T} \|z_l\| \, ds + R_2 \int_{t_0}^{T} \|z_l\| \, dV_{t_0}^{s} u.$$

Taking the supremum of both sides over all partitions of I, and then taking the limit of both sides as $l \to \infty$ we get

$$\lim_{l\to\infty} V_{t_0}^{T}(\varphi x_{k_l} - y) = 0. \tag{4.16}$$

Since $\|(\varphi(x_{k_l}) - y)\|_{BV} = V_{t_0}^{T}(\varphi(x_{k_l}) - y) + \left\|(\varphi(x_{k_l}) - y)(t_0^{+})\right\|$ we have

$$\lim_{l\to\infty} \|\varphi(x_{k_l}) - y\|_{BV} = 0,$$

i.e. the sequence $\{\varphi(x_k)\}$ contains a subsequence $\{\varphi(x_{k_l})\}$ which converges in $BV(I)$ to $y \in BV(I)$. This shows that the operator φ is compact. ■

Now, we are ready to establish our main results

4.2.2 Existence of Solutions

In this section we derive some existence and uniqueness results of solutions to system (4.5). The proof of these results relies on the topological transversality theorem and the Banach contraction fixed point theorem.

Theorem 4.8 *Assume that **(H1)** and **(H2)** hold. Then system (4.5) has at least one solution.*

Proof. It follows from Lemma 4.4 that any solution x of (4.5) is bounded in $BV(I)$. Hence, there is $r > 0$ such that

$$\|x\|_{BV} \leq r. \tag{4.17}$$

Let $\Omega = \{x \in BV(I) : \|x\|_{BV} < r+1\}$. Then Ω is an open bounded and convex subset of $BV(I)$. For $\lambda \in [0,1]$ define an operator $H(\lambda, \cdot) : \overline{\Omega} \to BV(I)$ by

$$H(\lambda, x)(t) = \lambda x_0 + \lambda \int_{t_0}^{t} f(s, x(s), u(s))ds + \lambda \int_{t_0}^{t} g(s, x(s), u(s))du(s),$$

Since $H(\lambda, \cdot) = \lambda\varphi(\cdot)$, with $\varphi(\cdot)$ given by (4.9), it follows that $H(\lambda, \cdot)$ is compact. It follows from (4.17) that $H(\lambda, \cdot)$ has no fixed point on $\partial\Omega$, and so it is an admissible homotopy ([53]) between the constant map $H(0, \cdot) \equiv 0$ and $H(1, \cdot) \equiv \varphi$. Since $H(0, \cdot)$ is essential then $H(1, \cdot)$ is essential which implies that $\varphi \equiv H(1, \cdot)$ has a fixed point in Ω. This fixed point is a solution of our problem. ■

Theorem 4.9 *Assume that, in addition to* ***(H1)*** *and* ***(H2)****, the following condition holds*

(H3) $(R_1 M A + 2R_2 R) < 1$, *where* $A = T - t_0$ *and* $R = V_{t_0}^{T} u$.

Then system (4.5) has a unique solution.

Proof. Any solution of (4.5) is a fixed point of the operator $\varphi(\cdot)$, defined by

$$\varphi(x(t)) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds + \int_{t_0}^{t} g(s, x(s), u(s))du(s).$$

Let $x, y \in BV(I)$. Then

$$\begin{aligned}
\left\|\varphi(x(t_0^+)) - \varphi(y(t_0^+))\right\| &= || \int_{t_0}^{t_0^+} [f(s, x(s), u(s)) - f(s, y(s), u(s))]ds + \\
&\quad \int_{t_0}^{t_0^+} [g(s, x(s), u(s)) - g(s, x(s), u(s))]du(s)||. \\
&\leq R_1 M \int_{t_0}^{t_0^+} \|x - y\|\, ds + R_2 \int_{t_0}^{t_0^+} \|x - y\|\, dV_{t_0}^{s} u \\
&\leq R_2 \|x - y\|\, V_{t_0}^{T} u.
\end{aligned}$$

Let $R = V_{t_0}^T u$. Since $\|x - y\| \leq \|x - y\|_{BV}$, we obtain

$$\left\|\varphi(x(t_0^+)) - \varphi(y(t_0^+))\right\| \leq R_2 R \|x - y\|_{BV}. \tag{4.18}$$

Let $\alpha_i, \alpha_j \in [t_0, T]$ with $\alpha_i < \alpha_j$. Then

$$\begin{aligned}
&\|[\varphi(x(\alpha_j)) - \varphi(y(\alpha_j))] - [\varphi(x(\alpha_i)) - \varphi(y(\alpha_i))]\| \\
&= \quad || \int_{\alpha_i}^{\alpha_j} [f(s, x(s), u(s)) - f(s, y(s), u(s))] ds \\
&\qquad\qquad + \int_{\alpha_i}^{\alpha_j} [g(s, x(s), u(s)) - g(s, x(s), u(s))] du(s)|| \\
&\leq \quad R_1 M \int_{\alpha_i}^{\alpha_j} \|x - y\| \, ds + R_2 \int_{\alpha_i}^{\alpha_j} \|x - y\| \, dV_{t_0}^s u.
\end{aligned}$$

For any partition $t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T$ of the interval I we obtain

$$\begin{aligned}
&\sum_{i=1}^{m} \|[\varphi(x(\alpha_i)) - \varphi(y(\alpha_i))] - [\varphi(x(\alpha_{i-1})) - \varphi(y(\alpha_{i-1}))]\| \\
&\leq \quad R_1 M \sum_{i=1}^{m} \int_{\alpha_i}^{\alpha_j} \|x - y\| \, ds + R_2 \sum_{i=1}^{m} \int_{\alpha_i}^{\alpha_j} \|x - y\| \, dV_{t_0}^s u \\
&\leq \quad R_1 M (T - t_0) \|x - y\|_{BV} + R_2 V_{t_0}^T u \|x - y\|_{BV}.
\end{aligned}$$

Taking supremum for both sides of the above inequality we get

$$V_{t_0}^T (\varphi(x) - \varphi(y)) \leq (R_1 M A + R_2 R) \|x - y\|_{BV}. \tag{4.19}$$

Combining (4.18) and (4.19) we get

$$\|\varphi(x) - \varphi(y)\|_{BV} \leq (R_1 M A + 2 R_2 R) \|x - y\|_{BV}.$$

Condition *(H3)* implies that φ is a contraction. By the Banach fixed point theorem φ has a unique fixed point x, which is the unique solution of (4.5). ■

4.2.3 Stability

In this section we are concerned with the stability of solutions of system (4.5).

Definition 4.8 *The system (4.5) is said to be stable, if given* (λ, θ) *with* $0 < \lambda < \theta$, *we have* $\|x_0\|_{BV} \leq \lambda$ *implies* $\|x\|_{BV} \leq \theta$.

Theorem 4.10 *Assume that the following conditions hold*

(i) $f : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is continuous in* (t, x) *for all* u *and satisfies*

$$\|f(t, x, u)\| \leq |u| \, \|x\| \, \alpha(t)$$

where $\alpha(t)$ *is a continuous function on* $[t_0, T]$.

(ii) $g : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is continuous in* (t, x) *for all* u *and satisfies*

$$\|g(t, x, u)\| \leq K \, |u| \, \|x\| \, .$$

(iii) $M(PA + 2KR) < 1$, *where* $P = \sup_{t \in I} |\alpha(t)|$.

Then system (4.1) is stable in the sense of Definition 4.8.

Proof. Let x be any solution of (4.5). Assume $\|x_0\|_{BV} \leq \lambda$ where $\lambda > 0$. Using (4.6) we have

$$x(t_0^+) = x_0 + \int_{t_0}^{t_0^+} f(s, x(s), u(s))ds + \int_{t_0}^{t_0^+} g(s, x(s), u(s))du(s).$$

Thus

$$\begin{aligned} \left\|x(t_0^+)\right\| &\leq \|x_0\| + \int_{t_0}^{t_0^+} \|f(s, x(s), u(s))\| \, ds + \int_{t_0}^{t_0^+} \|g(s, x(s), u(s))\| \, dV_{t_0}^s u \\ &\leq \|x_0\|_{BV} + M \, \|x\|_{BV} \int_{t_0}^{t_0^+} \alpha(s)ds + KM \, \|x\|_{BV} \, V_{t_0}^{t_0^+} u. \end{aligned}$$

Notice that $\|x(s)\| \leq \|x\|_{BV}$ for all $s \in I$. Since $\alpha(\cdot)$ is continuous on I we have

$$\left\|x(t_0^+)\right\| \leq \lambda + KMR\,\|x\|_{BV}\,, \tag{4.20}$$

where $V_{t_0}^T u = R$. Now, let $\alpha_i, \alpha_j \in [t_0, T]$ with $\alpha_i < \alpha_j$. Then

$$\begin{aligned}\|x(\alpha_j) - x(\alpha_i)\| &\leq \int_{\alpha_i}^{\alpha_j} \|f(s, x(s), u(s))\|\, ds + \int_{\alpha_i}^{\alpha_j} \|g(s, x(s), u(s))\|\, dV_{t_0}^s u \\ &\leq M\,\|x\|_{BV} \int_{\alpha_i}^{\alpha_j} \alpha(s) ds + KM\,\|x\|_{BV} \int_{\alpha_i}^{\alpha_j} dV_{t_0}^s u.\end{aligned}$$

Now, for any partition $t_0 = \alpha_0 < \alpha_1 < ... < \alpha_m = T$ of the interval I we obtain

$$\begin{aligned}\sum_{i=1}^{m} \|x(\alpha_i) - x(\alpha_{i-1})\| &\leq M\,\|x\|_{BV} \int_{t_0}^{T} \alpha(s) ds + KM\,\|x\|_{BV}\, V_{t_0}^T u \\ &\leq MPA\,\|x\|_{BV} + KMR\,\|x\|_{BV}\,,\end{aligned}$$

where $P = \sup_{t \in I} |\alpha(t)|$. Taking supremum of both sides we get

$$V_{t_0}^T x \leq M(PA + KR)\,\|x\|_{BV}\,. \tag{4.21}$$

From (4.20) and (4.21) we have

$$\|x\|_{BV} \leq \lambda + M(PA + 2KR)\,\|x\|_{BV}\,.$$

Then

$$(1 - M(PA + 2KR))\,\|x\|_{BV} \leq \lambda.$$

Since $1 - M(PA + 2KR) > 0$ we get

$$\|x\|_{BV} \leq \frac{1}{(1 - M(PA + 2KR))}\lambda := \theta.$$

Condition **(iii)** implies that $\theta > \lambda$. That is, there exist (λ, θ) such that $0 < \lambda < \theta$ and $\|x_0\|_{BV} \leq \lambda$ implies $\|x\|_{BV} \leq \theta$. This completes the proof of the theorem. ∎

Example 4.1 *Consider the system*

$$x' = xu \ , \ x(0) = x_0, \qquad t \in [0, T].$$

We know that $x(t) = x_0 e^{\int_0^t u(s)ds}$. *This leads to* $\|x\| \leq \|x_0\| e^{Mt} < \lambda e^{\bar{u}t} < \theta$ *where* $M = \sup_{t\in[o,T]} |u(t)|$ *and* $\theta = \lambda e^{MT}$.

Then this system is stable in the sense of Definition 4.8.

Example 4.2 *Consider the system*

$$x' = xu + xu \ u' \ , x(0) = x_0 \ , \qquad t \in [0, T].$$

Here we have that $x(t) = x_0 e^{\int_0^t u(s)ds + \frac{1}{2}u^2(t) - \frac{1}{2}u^2(0)}$, *so that*

$$\|x\| \leq \|x_0\| e^{Mt+M^2} < \lambda e^{Mt+M^2} < \theta,$$

where $\theta = \lambda e^{MT+M^2}$. *Then this system is stable using of Definition 4.8 by taking* $\theta = \lambda e^d$ *where* $d = MT + M^2 > 0$ *for any* $\lambda > 0$.

Remark 4.2 *If* $1 - M(PA + 2KR) = 0$ *then the system is stable in the sense of Definition 4.8, as seen from above Example 4.1.*

Remark 4.3 *The condition* $M(PA + 2KR) < 1$ *is too restrictive as seen in Example 4.2.*

Chapter 5

Parabolic Impulsive Differential Equations

5.1 Introduction

Many processes in the applied sciences experience abrupt changes in their states due to perturbations. The durations of these perturbations are negligeable in comparison with the duration of each process. It is natural to assume that the perturbations exhibit an impulsive behavior. A typical example is the modeling of the growth of a population diffusing through its habitat. The natural growth of the population is disturbed at some time intervals by, for instance, harvesting, or instantantaneous stocking. This process has been described by an impulsive initial-boundary value problem for a semilinear parabolic partial differential equation. The study of this type of problems started with the pionneer paper [46]. Almost all the works devoted to this problem have relied on the method of lower and upper solutions. See [19], [38]. In this chapter we shall study a simpler problem. We investigate the propagation of heat along a homogeneous rod of length A under the influence of a nonlinear heat source. We assume that the rod is placed along the x-axis. Let $u\left(x,t\right)$ represent the temperature along the rod at position x and time t. We will assume that the initial temperature along the rod to be $u(x,0)=0,\ x\in\left[0,A\right]$. At time $t_i,\ \ i=1,2,...,m$ in $(0,T)$ with $0<t_1<...<t_m<T$, called impulse moments, another nonlinear heat source is applied at (x,t_i). The mathematical model is described by the following impulsive parabolic problem (see [91] for a corresponding linear model). Let $D=(0,A)\times(0,T)$ and $P_j=\{(x,t_j);\ x\in(0,A)\},\ P=\cup_{j=1}^{m}P_j$. Our aim is to investigate the existence of solutions of the following nonlinear impulsive parabolic problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}+f(x,t,u),\quad x\in D\setminus P, \tag{5.1}$$

$$u(x,0) = 0,\qquad x\in\left[0,A\right], \tag{5.2}$$

$$u(0,t) = u(A,t)=0,\qquad t\in\left[0,T\ \right), \tag{5.3}$$

$$u(x,t_i^{+}) = u(x,t_i^{-})+\sigma_i\left(x,t_i^{-},u\right),\quad (x,t_i)\in P_i,\ i=1,2,...,m. \tag{5.4}$$

For the sake of simplicity of the presentation we shall study the case of only one impulse moment, i.e. $m=1$. Our approach shall be based on Green's function and fixed point theorems in appropriate Banach spaces.

5.2 Preliminaries

For $u:D\to\mathbb{R}$ we denote its partial derivatives (when they exist) by $D_tu=\partial u/\partial t$, $D_xu=\partial u/\partial x,\ D_{xx}u=\partial^2u/\partial x^2$.

$C(D)$ denotes the Banach space of continuous functions $u:D\to\mathbb{R}$, endowed with the norm

$$\left|u\right|_0=\sup\{\left|u(x,t)\right|;\ (x,t)\in\overline{D}\}.$$

We say that $u \in C^{2,1}(D)$ if $u,\ D_x u,\ D_{xx}u$ and $D_t u$ exist and are continuous on D. In fact, we can write

$$C^{2,1}(D) = \{u \in C(D);\ u(.,t) \in C^2(0,A), t \in (0,T)\,,\ u(x,.) \in C^1\,(0,T)\,, x \in (0,A)\}.$$

$u \in C(D)$ is called Hölder continuous of order $\alpha \in (0,1]$ if

$$H_\alpha\,(u) = \sup\{\frac{|u(x,t) - u(\xi,\tau)|}{\left(\|x-\xi\|^2 + |t-\tau|\right)^{\alpha/2}}; (x,t)\,, (\xi,\tau) \in D\} < +\infty.$$

In this case we write $u \in C^\alpha(D)$ and we define its norm by

$$|u|_\alpha = |u|_0 + H_\alpha\,(u)\,.$$

If $\alpha = 1$, u is called Lipschitz continuous. Note that the natural injection i : $C^\alpha(D) \to C(D)$ is continuous. We say that that $C^\alpha(D)$ is continuously embedded in $C(D)$, and we write $C^\alpha(D) \hookrightarrow C(D)$.

Also, $u \in C^{2+\alpha,1+\alpha}(D)$ if $u(.,t) \in C^{2+\alpha}(0,A)$ for all $t \in (0,T)$ and $u(x,.) \in C^{1+\alpha}(0,T)$ for all $x \in (0,A)$. For $u \in C^{2+\alpha,1+\alpha}(D)$ we define its norm by

$$|u|_{2+\alpha,1+\alpha} = |u|_\alpha + |D_x u|_\alpha + |D_{xx}u|_\alpha + |D_t u|_\alpha\,.$$

Next, we introduce the Lebesgue and Sobolev spaces. For $1 \leq p < +\infty$, we say that $u : (0,A) \to \mathbb{R}$ is in $L^p(0,A)$ if u is measurable and $\int_0^A |u(x)|^p\,dx < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_0^A |u(x)|^p\,dx\right)^{1/p}.$$

For $p = +\infty$, we write

$$\begin{aligned} |u|_\infty &= ess\sup\{|u(x)|\,; x \in (0,A)\} \\ &= \inf_{N \subset (0,A), \mu(N)=0}\ \sup_{x \in (0,A)\setminus N} |u(x)|\,,\ \mu = \text{Lebesgue measure.} \end{aligned}$$

We have the natural embeddings $L^q(0,A) \subset L^p(0,A)$ for $p < q$. In particular, $L^2(0,A) \subset L^1(0,A)$, for, it is clear that $|u|_{L^1} \leq |u|_{L^2}\ A^{1/2}$.

5.3 Linear Impulsive Parabolic Problem

Now, we turn to the study of the linear impulsive parabolic problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t), \quad (x,t) \in D \setminus P_1\ , \tag{5.5}$$

$$u(x,0) = 0, \qquad x \in [0,A]\,, \tag{5.6}$$

$$u(0,t) = u(A,t) = 0, \qquad t \in [0,T\ ], \tag{5.7}$$

$$u(x,t_1^+) = u(x,t_1^-) + \theta\,(x,t_1)\,, \quad (x,t_1) \in P_1. \tag{5.8}$$

The following notations have been introduced in [46]. $C^{2,1}(D \backslash P_1)$ is the set of all functions $u : \overline{D} \to \mathbb{R}$ satisfying the following

(i) $u(x,\cdot) \in C^1\left(\overline{D}\ \backslash P_1\right)$,

(ii) $u(\cdot,t) \in C^2\left(\overline{D}\ \backslash P_1\right)$,

(iii) for $w = (u, D_t u, D_x u,\ D_{xx} u)$ let

$$\lim_{(y,s)\to\left(x,t_1^-\right)} w(y,s) = w(x,t_1),$$

and $\lim_{(y,s)\to\left(x,t_1^+\right)} w(y,s)$ exists for $x \in [0,A]$.

Definition 5.1 *$u \in C^{2,1}(D \backslash P_1)$ satisfying (5.5) - (5.8) is called a solution of the impulsive initial-boundary value problem for the parabolic equation.*

Lemma 5.1 *Problem (5.5) - (5.8) has a unique solution $u \in C^{2,1}(D \backslash P_1)$ given by*

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds \\ &\quad + \int_0^A G(x, t-t_1, \xi)\theta(\xi, t_1) d\xi. \end{aligned} \tag{5.9}$$

Remark 5.1 *It is understood that for $t < t_1$, $\int_0^A G(x, t-t_1, \xi)\theta(\xi, t_1)d\xi = 0$.*

Remark 5.2 *The integral representation $\int_0^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds$ is well known. For more details we refer the interested reader to the books [48], [70], [98].*

Proof. It is well known (see for instance [129]), that the eigenvalue problem

$$z"(x) + \lambda z(x) = 0, \quad z(0) = z(A) = 0,$$

has a sequence of distinct eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{A^2}, \ n = 1, 2, ...,$$

and a sequence of corresponding orthogonal eigenfunctions

$$z_n(x) = \sin \frac{n\pi}{A} x, \quad n = 1, 2, ...$$

We shall look for a solution $u(x,t)$ of the linear problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t), & (x,t) \in (0,A) \times (0,T), \\ u(x,0) = 0, & x \in [0, A], \\ u(0,t) = u(A,t) = 0, & t \geq 0, \end{cases}$$

in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{A} x.$$

Substituting in the parabolic partial differential equation we see that

$$u_n'(t) + \lambda_n u_n(t) = g_n(t), \quad u_n(0) = 0, \tag{5.10}$$

where $g_n(t) = \frac{2}{A}\int_0^A g(\xi, t) \sin \frac{n\pi}{A}\xi d\xi$.

It follows from (5.10) that

$$u_n(t) = \int_0^t e^{-\lambda_n(t-s)} g_n(s) ds.$$

Substituting the expression of $g_n(t)$ we obtain

$$\begin{aligned} u_n(t) &= \int_0^t e^{-\lambda_n(t-s)} \left(\frac{2}{A} \int_0^A g(\xi, s) \sin \frac{n\pi}{A}\xi d\xi \right) ds \\ &= \int_0^t \int_0^A \frac{2}{A} e^{-\lambda_n(t-s)} g(\xi, s) \sin \frac{n\pi}{A}\xi d\xi ds. \end{aligned}$$

Thus

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{A} x$$

becomes

$$u(t, x) = \sum_{n=1}^{\infty} \left[\int_0^t \int_0^A \frac{2}{A} e^{-\lambda_n(t-s)} g(\xi, s) \sin \frac{n\pi}{A}\xi d\xi ds \right] \sin \frac{n\pi}{A} x.$$

Notice that the series $\sum\limits_{n=1}^{\infty} e^{-\lambda_n(t-s)} \sin \frac{n\pi}{A}\xi \sin \frac{n\pi}{A} x$ is uniformly convergent for $t > s$. This will allow us to write

$$\begin{aligned} u(t, x) &= \int_0^t \int_0^A \left[\frac{2}{A} \sum_{n=1}^{\infty} e^{-\lambda_n(t-s)} \sin \frac{n\pi}{A}\xi \sin \frac{n\pi}{A} x \right] g(\xi, s) d\xi ds \\ &= \int_0^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds, \end{aligned}$$

where

$$G(x, t-s, \xi) = \frac{2}{A} \sum_{n=1}^{\infty} e^{-\lambda_n(t-s)} \sin \frac{n\pi}{A}\xi \sin \frac{n\pi}{A} x$$

is the Green's function given by its bilinear expantion.

Hence the solution of the problem (5.5), (5.6), (5.7) is given by

$$u(x,t) = \int_0^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds. \tag{5.11}$$

The solution on $(0, A) \times (0, t_1)$ is given by (see (5.11))

$$u(x,t) = \int_0^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds, \quad x \in (0, A), \quad t \in (0, t_1). \tag{5.12}$$

The solution on $(0, A) \times (t_1, t)$ is given by

$$\begin{aligned} u(x,t) &= \int_{t_1}^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds + \int_0^A G(x, t-t_1, \xi) u(\xi, t_1) d\xi \\ &\quad + \int_0^A G(x, t-t_1, \xi) \theta(\xi, t_1) d\xi \\ &= \int_{t_1}^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds \\ &\quad + \int_0^A G(x, t-t_1, \xi) \left(\int_0^{t_1} \int_0^A G(\xi, t_1 - s, \eta) g(\eta, s) d\eta ds \right) d\xi \\ &\quad + \int_0^A G(x, t-t_1, \xi) \theta(\xi, t_1) d\xi \\ &= \int_{t_1}^t \int_0^A G(x, t-s, \xi) g(\xi, s) d\xi ds \\ &\quad + \int_0^{t_1} \int_0^A \left(\int_0^A G(x, t-t_1, \xi) G(\xi, t_1 - s, \eta) d\xi \right) g(\eta, s) d\eta ds \\ &\quad + \int_0^A G(x, t-t_1, \xi) \theta(\xi, t_1) d\xi. \end{aligned}$$

Using the orthogonality property of the eigenfunctions $\sin \frac{n\pi}{A} x, \ n = 1, 2, ...,$ we can easily show that

$$\int_0^A G(x, t-t_1, \xi) G(\xi, t_1 - s, \eta) d\xi = G(x, t-s, \eta).$$

Hence

$$
\begin{aligned}
u(x,t) &= \int_0^{t_1}\int_0^A G(x,t-s,\xi)g(\xi,s)d\xi ds \\
&+ \int_{t_1}^{t}\int_0^A G(x,t-s,\eta)g(\eta,s)d\eta ds \\
&+ \int_0^A G(x,t-t_1,\xi)\theta(\xi,t_1)d\xi.
\end{aligned}
$$

Finally, we see that

$$
\begin{aligned}
u(x,t) &= \int_0^{t}\int_0^A G(x,t-s,\xi)g(\xi,s)d\xi ds \\
&+ \int_0^A G(x,t-t_1,\xi)\theta(\xi,t_1)d\xi,
\end{aligned} \tag{5.13}
$$

is the solution of (5.5), (5.6), (5.7), (5.8). ■

5.4 Nonlinear Impulsive Parabolic Problems

The following result is a direct consequence of the previous section

Proposition 1 *$u \in C^{2,1}(D\backslash P_1)$ is a solution of (5.1), (5.2), (5.3), (5.4) if and only if $u \in C(D\backslash P_1)$ is a solution of the nonlinear integral equation*

$$
\begin{aligned}
u(x,t) &= \int_0^{t}\int_0^A G(x,t-s,\xi)f(\xi,s,u(\xi,s))d\xi ds \\
&+ \int_0^A G(x,t-t_1,\xi)\sigma_1(\xi,t_1,u(\xi,t_1))d\xi.
\end{aligned} \tag{5.14}
$$

For $u \in C(D\backslash P_1)$ we define its norm by

$$|u|_0 = \sup\{|u(x,t)|\,;(x,t) \in \bar{D}\}.$$

Then $(C(D\backslash P_1), |\cdot|_0)$ is a Banach space.

Define an operator $\psi : C(D\backslash P_1) \to C(D\backslash P_1)$ by

$$(\psi u)(x,t) = \text{ the right-hand side of (5.14).}$$

We introduce the followong conditions

(H1) $f(\cdot,\cdot,u) : D \to \mathbb{R}$ is Hölder continuous and there exists $l : D \to \mathbb{R}_+$, Hölder continuous such that

$$|f(x,t,u) - f(x,t,v)| \leq l(x,t)\,|u-v|\,,$$

(H2) $\sigma_1(\cdot,\cdot,u) : D \to \mathbb{R}$ is Hölder continuous and there exists $\eta : [0,A] \to \mathbb{R}_+$, continuous such that

$$|\sigma_1(x,t,u) - \sigma_1(x,t,v)| \leq \eta(x)\,|u-v|\,,$$

(H3) $\sup\limits_{(x,t)\in\bar{D}} \int_0^t \int_0^A |G(x,t-s,\xi)|\, l(\xi,s) d\xi ds +$

$$\sup_{(x,t)\in\bar{D}} \int_0^A |G(x,t-t_1,\xi)|\,\eta(\xi) d\xi < 1.$$

Theorem 5.1 *Suppose that **(H1)**, **(H2)**, **(H3)** are satisfied. Then the impulsive initial-boundary value problem (5.1), (5.2), (5.3), (5.4) has a unique solution.*

Proof. We shall show that the nonlinear operator ψ is contraction.

For any $(x,t) \in D$, we have

$$\begin{aligned}\psi_u(x,t) - \psi_v(x,t) &= \int_0^t \int_0^A G(x,t-s,\xi)\,[f(\xi,s,u(\xi,s)) - f(\xi,s,v(\xi,s))]\,d\xi ds \\ &\quad + \int_0^A G(x,t-t_1,\xi)\,[\sigma_1(\xi,t_1,u(\xi,t_1)) - \sigma_1(\xi,t_1,v(\xi,t_1))]\,d\xi.\end{aligned}$$

This implies that

$$
\begin{aligned}
|\psi_u(x,t)-\psi_v(x,t)| &\le \int_0^t\int_0^A |G(x,t-s,\xi)|\, l(\xi,s,)\, |u(\xi,s)-v(\xi,s)|\, d\xi ds \\
&\quad + \int_0^A |G(x,t-t_1,\xi)|\, \eta(\xi)\, |u(\xi,t_1)-v(\xi,t_1)|\, d\xi \\
&\le \left\{ \int_0^t\int_0^A |G(x,t-s,\xi)|\, l(\xi,s,) d\xi ds \right. \\
&\quad \left. + \int_0^A |G(x,t-t_1,\xi)|\, \eta(\xi) d\xi \right\} |u-v|_0 .
\end{aligned}
$$

Condition (H3) implies that

$$
|\psi_u(x,t)-\psi_v(x,t)| \le \gamma\, |u-v|_0 ,
$$

where

$$
\begin{aligned}
\gamma &= \sup_{(x,t)\in\bar{D}} \int_0^t\int_0^A |G(x,t-s,\xi)|\, l(\xi,s) d\xi ds \\
&\quad + \sup_{(x,t)\in\bar{D}} \int_0^A |G(x,t-t_1,\xi)|\, \eta(\xi) d\xi.
\end{aligned}
$$

Hence ψ is a contraction. By the Banach contraction principle ψ has a unique fixed point $u \in C(D\backslash P_1)$, which is the unique solution of the integral equation (5.14). This, in turn, implies that (5.1), (5.2), (5.3), (5.4) has a unique solution. ■

Our second result is based on the Schauder Nonlinear Alternative (see [53]). We need the following assumptions:

(H$_f$) $\cdot\ f(\cdot,\cdot,u) : D \to \mathbb{R}$ is Hölder continuous,

$\cdot\ f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous,

· there exist $p : D \to \mathbb{R}$ Hölder continuous, $l_0 : \mathbb{R} \to \mathbb{R}$ continuous and nondecreasing such that

$$|f(x,t,u)| \le p(x,t) l_0(|u|).$$

(H$_\sigma$) · $\sigma_1(\cdot,\cdot,u) : D \to \mathbb{R}$ is Hölder continuous,

· $\sigma_1(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous,

· there exist $q : [0,A] \to \mathbb{R}$ continuous, $l_1 : \mathbb{R} \to \mathbb{R}$ continuous and nondecreasing such that

$$|\sigma_1(x,t_1,u)| \le q(x) l_1(|u|).$$

(H$_{f,\sigma}$) $\sup\limits_{\rho>0} \frac{\rho}{|G_p|_0 l_0(\rho) + |G_q|_0 l_1(\rho)} > 1,$

where

$$|G_p|_0 = \sup_{(x,t)\in\bar{D}} \int_0^t \int_0^A |G(x,t-s,\xi)| \, p(\xi,s) d\xi ds,$$

and

$$|G_q|_0 = \sup_{(x,t)\in\bar{D}} \int_0^A |G(x,t-t_1,\xi)| \, q(\xi) d\xi.$$

Theorem 5.2 *If the conditions **(H$_f$)**, **(H$_\sigma$)**, **(H$_{f,\sigma}$)** are satisfied then problem (5.1), (5.2), (5.3), (5.4) has at least one solution.*

Proof. Let ψ be the operator defined previously, i.e.

$$\begin{aligned} \psi u(x,t) &= \int_0^t \int_0^A G(x,t-s,\xi) f(\xi,s,u(\xi,s)) d\xi ds \\ &\quad + \int_0^A G(x,t-t_1,\xi)\sigma_1(\xi,t_1,u(\xi,t_1)) d\xi. \end{aligned}$$

We see that

(i) ψ maps bounded subset of $C(D\backslash P_1)$ into bounded subsets. For, let M be a bounded subset of $C(D\backslash P_1)$. Then, there exists $r > 0$ such that

$$|u|_0 \leq r \quad \text{for all } u \in M.$$

We have

$$\begin{aligned}
|\psi u(x,t)| &\leq \int_0^t \int_0^A |G(x,t-s,\xi)|\, p(\xi,s) l_0(|u(\xi,s)|) d\xi ds \\
&\quad + \int_0^A |G(x,t-t_1,\xi)|\, q(\xi) l_1(|u(\xi,t_1)|) d\xi \\
&\leq \left(\int_0^t \int_0^A |G(x,t-s,\xi)|\, p(\xi,s) d\xi ds \right) l_0(r) \\
&\quad + \left(\int_0^A |G(x,t-t_1,\xi)|\, q(\xi) d\xi \right) l_1(r) \\
&\leq |G_p|_0\, l_0(r) + |G_q|_0\, l_1(r).
\end{aligned}$$

Hence

$$|\psi u|_0 \leq R := |G_p|_0\, l_0(r) + |G_q|_0\, l_1(r).$$

(ii) ψ maps bounded subsets of $C(D\backslash P_1)$ into equicontinuous substs.

Let $u \in M$. Then $|u|_0 \leq r$.

Also, let $x, y \in [0, A]$ and $t, \tau \in (0, T]$ with $t > \tau > 0$. Then

$$
\begin{aligned}
\psi u(x,t) - \psi u(y,\tau) &= \int_0^t \int_0^A G(x, t-s, \xi) f(\xi, s, u(\xi, s)) d\xi ds \\
&\quad + \int_0^A G(x, t-t_1, \xi)\sigma_1(\xi, t_1, u(\xi, t_1)) d\xi \\
&\quad - \int_0^\tau \int_0^A G(y, \tau - s, \xi) f(\xi, s, u(\xi, s)) d\xi ds \\
&\quad - \int_0^A G(y, \tau - t_1, \xi)\sigma_1(\xi, t_1, u(\xi, t_1)) d\xi \\
&= \int_0^\tau \int_0^A \left(G(x, t-s, \xi) - G(y, \tau - s, \xi)\right) f(\xi, s, u(\xi, s)) d\xi ds \\
&\quad + \int_\tau^t \int_0^A G(x, t-s, \xi) f(\xi, s, u(\xi, s)) d\xi ds \\
&\quad + \int_0^A \left(G(x, t-t_1, \xi) - G(y, \tau - t_1, \xi)\right) \sigma_1(\xi, t_1, u(\xi, t_1)) d\xi.
\end{aligned}
$$

Recall that

$$G(x, t-s, \xi) = \frac{2}{A} \sum_{n=1}^{\infty} e^{-\lambda_n (t-s)} \sin \frac{n\pi}{A} x \sin \frac{n\pi}{A} \xi,$$

with $\lambda_n = \frac{n^2\pi^2}{A^2}$, $n = 1, 2, 3, \ldots$.

It follows that

$$G(x, t-s, \xi) - G(y, \tau - s, \xi) = \frac{2}{A} \sum_{n=1}^{\infty} \left(e^{-\lambda_n (t-s)} \sin \frac{n\pi}{A} x - e^{-\lambda_n (\tau - s)} \sin \frac{n\pi}{A} y \right) \sin \frac{n\pi}{A} \xi.$$

Define a function $h_n : D \to \mathbb{R}$ by

$$h_n(x, t-s) = e^{-\lambda_n (t-s)} \sin \frac{n\pi}{A} x, \;\; n = 1, 2, 3, \ldots.$$

Then h_n is continuously differentiable and

$$G(x, t-s, \xi) - G(y, \tau - s, \xi) = \frac{2}{A} \sum_{n=1}^{\infty} (h_n(x, t-s) - h_n(y, \tau - s)) \sin \frac{n\pi}{A} \xi.$$

The mean value theorem for functions of two variables imply

$$h_n(x,t-s) - h_n(y,\tau-s) = \frac{\partial h_n}{\partial x}(z,\mu)\cdot(x-y) + \frac{\partial h_n}{\partial t}(z,\mu)\cdot(t-\tau),$$

where $z = y + \theta(x-y)$, $\mu = \tau + \theta(t-\tau)$ and $0 < \theta < 1$.

Now,

$$\frac{\partial h_n}{\partial x}(z,\mu) = \frac{n\pi}{A} e^{-\lambda_n \mu} \cos\frac{n\pi}{A} z,$$

and

$$\frac{\partial h_n}{\partial t}(z,\mu) = -\lambda_n e^{-\lambda_n \mu} \sin\frac{n\pi}{A} z.$$

Hence

$G(x,t-s,\xi) - G(y,\tau-s,\xi) =$

$$\left(\frac{2\pi}{A^2}\sum_{n=1}^{\infty}\left(ne^{-\lambda_n(\tau-s+\theta(t-\tau))}\cos\frac{n\pi}{A}(y+\theta(x-y))\right)\sin\frac{n\pi}{A}\xi\right)(x-y)$$
$$+\left(\frac{2}{A}\sum_{n=1}^{\infty}\left((-\lambda_n)e^{-\lambda_n(\tau-s+\theta(t-\tau))}\sin\frac{n\pi}{A}(y+\theta(x-y))\right)\sin\frac{n\pi}{A}\xi\right)(t-\tau),$$

Therefore

$$|G(x,t-s,\xi) - G(y,\tau-s,\xi)| \leq \frac{2}{A}\sum_{n=1}^{\infty}\left(\frac{n\pi}{A}|x-y| + \lambda_n|t-\tau|\right)e^{-\lambda_n(\tau-s+\theta(t-\tau))}.$$

Similarly, we have

$$G(x,t-t_1,\xi) - G(y,\tau-t_1,\xi) \leq \frac{2}{A}\sum_{n=1}^{\infty}\left(\frac{n\pi}{A}|x-y| + \lambda_n|t-\tau|\right)e^{-\lambda_n(\tau-t_1+\theta(t-\tau))}.$$

Next, it follows from $(\mathbf{H}_f)$ and $(\mathbf{H}_\sigma)$ that

$$|\psi u(x,t) - \psi u(y,\tau)| \leq$$

$$\left(\int_0^\tau \int_0^A |G(x,t-s,\xi) - G(y,\tau-s,\xi)|\, d\xi ds\right) |p|_0\, l_0(r)$$
$$+\left(\int_\tau^t \int_0^A |G(x,t-s,\xi)|\, d\xi ds\right) |p|_0\, l_0(r)$$
$$+\left(\int_0^A |G(x,t-t_1,\xi) - G(y,\tau-t_1,\xi)|\, d\xi\right) |p|_0\, l_1(r).$$

The above inequality leads to

$$|\psi u(x,t) - \psi u(y,\tau)| \leq$$

$$\left(\int_0^\tau \int_0^A \frac{2}{A}\sum_{n=1}^{\infty}\left(\frac{n\pi}{A}|x-y| + \lambda_n |t-\tau|\right) e^{-\lambda_n(\tau-s+\theta(t-\tau))} d\xi ds\right) |p|_0\, l_0(r)$$
$$+\left(\int_\tau^t \int_0^A |G(x,t-s,\xi)|\, d\xi ds\right) |p|_0\, l_0(r)$$
$$+\left(\int_0^A \frac{2}{A}\sum_{n=1}^{\infty}\left(\frac{n\pi}{A}|x-y| + \lambda_n |t-\tau|\right) e^{-\lambda_n(\tau-t_1+\theta(t-\tau))} d\xi\right) |p|_0\, l_1(r).$$

A simple computation gives

$$\int_0^\tau e^{-\lambda_n(\tau-s+\theta(t-\tau))} ds \leq \frac{1}{\lambda_n} e^{-\lambda_n\theta(t-\tau)}.$$

Since $\frac{n\pi}{A} = \sqrt{\lambda_n}$ the first term on the right hand side of the above inequality is bounded by

$$2\sum_{n=1}^{\infty}\left(\frac{|x-y|}{\sqrt{\lambda_n}} + |t-\tau|\right) e^{-\lambda_n\theta(t-\tau)} |p|_0\, l_0(r) =$$
$$\left(2\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}} e^{-\lambda_n\theta(t-\tau)} |p|_0\, l_0(r)\right) |x-y|$$
$$+\left(2\sum_{n=1}^{\infty} e^{-\lambda_n\theta(t-\tau)} |p|_0\, l_0(r)\right) |t-\tau|.$$

To estimate the third term notice that

$$\tau - t_1 + \theta(t-\tau) \geq \tau - t_1,$$

so that

$$e^{-\lambda_n(\tau-t_1+\theta(t-\tau))} \le e^{-\lambda_n(\tau-t_1)}.$$

Hence the third term is bounded by

$$\left(2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n(\tau-t_1)}}{\sqrt{\lambda_n}}\,|p|_0\,l_1(r)\right)|x-y| + \left(2\sum_{n=1}^{\infty}\lambda_n e^{-\lambda_n(\tau-t_1)}\,|p|_0\,l_1(r)\right)|t-\tau|\,.$$

Finally, we have

$$\begin{aligned}\int_{\tau}^{t}\int_{0}^{A}|G(x,t-s,\xi)|\,d\xi ds &\le \int_{\tau}^{t}\int_{0}^{A}\frac{2}{A}\sum_{n=1}^{\infty}e^{-\lambda_n(t-s)}d\xi ds\\ &\le 2\sum_{n=1}^{\infty}\int_{\tau}^{t}e^{-\lambda_n(t-s)}ds\\ &\le \sum_{n=1}^{\infty}\frac{2}{\lambda_n}\left(1-e^{-\lambda_n(t-\tau)}\right).\end{aligned}$$

It is clear that the following series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}e^{-\lambda_n\theta(t-\tau)} \quad \text{and} \quad \sum_{n=1}^{\infty}e^{-\lambda_n\theta(t-\tau)},$$

are uniformly convergent for $t>\tau$.

Also,

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}e^{-\lambda_n(\tau-t_1)} \quad \text{and} \quad \sum_{n=1}^{\infty}\lambda_n e^{-\lambda_n(\tau-t_1)},$$

are uniformly convergent for $\tau>t_1$.

Similarly

$$\sum_{n=1}^{\infty}\frac{2}{\lambda_n}\left(1-e^{-\lambda_n(t-\tau)}\right)$$

is uniformly convergent for $t > \tau$.

It follows from the above discussion and inequalities that

$$\psi u(x,t) - \psi u(y,\tau) \to 0 \quad \text{as} \quad (x,t) \to (y,\tau).$$

This shows that $\{\psi u; |u|_0 \leq r\}$ is equicontinuous. Consequently, Ascoli-Arzela theorem implies that $\overline{\psi(M)}$ is compact. Hence the operator ψ is completely continuous.

To complete the proof of the theorem, we show that the set of solutions u of $u = \lambda\psi u,\ 0 < \lambda < 1$, is bounded.

To see this, let u be any solution of

$$u = \lambda\psi u,\ 0 < \lambda < 1.$$

Then

$$\begin{aligned} u(x,t) &= \lambda \int_0^t \int_0^A G(x,t-s,\xi) f(\xi,s,u(\xi,s)) d\xi ds \\ &\quad + \lambda \int_0^A G(x,t-t_1,\xi)\sigma_1(\xi,t_1,u(\xi,t_1))d\xi, \end{aligned}$$

so that

$$\begin{aligned} |u(x,t)| &\leq \int_0^t \int_0^A |G(x,t-s,\xi)|\, p(\xi,s) l_0(|u(\xi,s)|) d\xi ds \\ &\quad + \int_0^A |G(x,t-t_1,\xi)|\, q(\xi) l_1(|u(\xi,t_1)|) d\xi. \end{aligned}$$

Let $\rho_0 = |u|_0$. Then

$$\begin{aligned} \rho_0 &\leq \left(\sup_{(x,t)\in D} \int_0^t \int_0^A |G(x,t-s,\xi)|\, p(\xi,s) d\xi ds \right) l_0(\rho_0) \\ &\quad + \left(\sup_{(x,t)\in D} \int_0^A |G(x,t-t_1,\xi)|\, q(\xi) d\xi \right) l_1(\rho_0) \\ &\leq |G_p|_0\, l_0(\rho_0) + |G_q|_0\, l_1(\rho_0). \end{aligned}$$

Hence

$$\frac{\rho_0}{|G_p|_0\, l_0(\rho_0) + |G_q|_0\, l_1(\rho_0)} \leq 1. \tag{5.15}$$

Condition ($\mathrm{H}_{f,\sigma}$) implies that there exists $\rho^* > 0$ such that for all $\rho > \rho^*$ we have

$$\frac{\rho}{|G_p|_0\, l_0(\rho) + |G_q|_0\, l_1(\rho)} > 1. \tag{5.16}$$

Comparing inequalities (5.15) and (5.16) we see that $\rho_0 \leq \rho^*$.

Consequently, we have shown that all solutions of $u = \lambda\psi u,\ 0 < \lambda < 1$, satisfy

$$|u|_0 \leq \rho^*.$$

It follows from Schauder Nonlinear Alternative theorem that ψ has a fixed point. This fixed point is a solution to our original problem. This completes the proof of the theorem. ■

Chapter 6

Concluding Remarks

Our main objectives in this thesis were to address the problem of existence, uniqueness and stability of solutions of impulsive differential systems and impulsive control systems. Three major areas were invstigated:

- second order impulsive control systems with boundary conditions

$$\begin{cases} x''(t) = F(t, x(t), x'(t)), \ t \in [0, T] \backslash \{t_1, t_2, ..., t_m\}, \\ \Delta x(t_k) = U_k(x(t)), k = 1, 2, ..., m, \\ \Delta x'(t_k) = V_k(x'(t)), \ k = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases}$$

- impulsive control systems with controls as functions of bounded variation

$$\begin{cases} x'(t) = f(t, x(t), u(t)) + g(t, x(t), u(t))u'(t), \quad t \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

- parabolic equations with impulsive effects

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(x,t,u(x,t)),\ t \in [0,T\ ]\backslash\{t_1\}, \quad x \in [0,A]\,, \\
u(x,0) &= 0, \qquad x \in [0,A]\,, \\
u(0,t) &= u(A,t) = 0, \qquad t \in [0,T\ ), \\
u(x,t_1^+) &= u(x,t_1^-) + \sigma_1\left(x,t_1^-,u(x,t_1^-)\right).
\end{aligned}$$

We have obtained several interesting results summarized in the following publications/ prepublications list:

(i) Existence of solutions for second order impulsive boundary value problems, Elect. J. Diff. Equ. Vol. 2012, No. 24 (2012), 10pp.

(ii) Impulsive control systems corresponding to controls of bounded variation, (under review).

(iii) Existence of solutions for second order impulsive control problems with boundary conditions, (under review).

(iv) Parabolic equations with impulsive effects, (under preparation).

Suggestions for Future Research

It is worthwile to consider the

- use of second method of Lyapunov to study the stability of impulsive control systems,
- investigation of parabolic impulsive problems with general second order parabolic operators,
- study of impulsive evolution equations.

References

[1] Ahmad, B., Existence of solutions for second-order nonlinear impulsive boundary-value problems, Elect. J. Diff. Equ. 2009 (2009), 1–7.

[2] Ahmed N. U., Existence of Optimal Controls for a General Class of Impulsive Systems on Banach Spaces, SIAM J. Control Optim. 42, No. 2 (2003), 669-685.

[3] Akhmetov, M. U. and Sejilova, R., The control of the Boundary-Value Problem for Linear Impulsive Integro-Differential Systems, J. Math. Anal. Appl. 236 No. 2 (1999), 312-326.

[4] Akhmetov, M. U. and Zafer, A., Controllability of the Vallée-Poussin Problem for Impulsive Differential Systems, J. Optim. Theo. Appl. 102, No. 2(1999), 263-276.

[5] Akhmetov, M. U. and Zafer, A., The controllability of Boundary-Value Problems for Quasilinear Impulsive Systems, Nonlinear Analysis 34, No. 7 (1998), 1055-1065.

[6] Akhmetov, M. U., Perestyuk, N. A. and Tieubergenova, M. A., The control of Linear Impulsive Systems, Ukrain. Math. Zh. 47(1995), 307-314.

[7] Akinyele, O., Cone-valued Lyapunov Functions and Stability of Impulsive Control Systems, Non. Analy. 39(2000), 247-259.

[8] Bai, C. and Yang, D., Existence of solutions for second-order nonlinear impulsive differential equations with periodic boundary value conditions, Hindawi Publishing Corporation BVPs. 2007, Article ID 41589, 13 pages.

[9] Bailey, P. B., Shampine, L. F. and Waltman, P. E., Nonlinear Two Point Boundary Value Problems, Academic Press, New York and London, 1968.

[10] Bainov, D. and Minchev, E., Forced oscillations of solutions of impulsive nonlinear parabolic differential-difference equations, J. Korean Math. Soc. 35 (1998), 881-890.

[11] Bainov, D. and Minchev, E., On the periodic boundary value problem for impulsive perabolic equations, Quarterly Appl. Math., VOl. LVII, No. 3 (1999), 543-547.

[12] Bainov, D., Kolev, D. and Motreanu, D., Blow-up of the solutions for an impulsive model with nonlinear boundary condition, PanAmerican Math. J., 13 (2003), 67-81.

[13] Bainov D. D., Kolev, D. A. and Nakagawa, K, Solutions with polynomial growth to an impulsive parabolic model, PanAmerican Math. J., 10 (2000), 47-57.

[14] Bainov, D. and Minchev, E., Oscillation of the solutions of impulsive parabolic equations, J.Comput. App. Math. 69 (1996), 207-214.

[15] Bainov, D. D. and Hristova, S. G., Impulsive integral inequalities with a deviation of the argument, Math. Nachr. 171 (1995), 19-27.

[16] Bainov, D. D. and Simeonov, P. S., Impulsive Differential Dquations: Periodic Solutions and Applications, Longman Scientific and Technical, Essex, England, 1993.

[17] Bainov, D. D. and Simeonov, P. S., Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific, Singapore, 1995.

[18] Bainov, D. D., Kamont, Z. and Minchev, E., Initial-boundary value problems for impulsive parabolic functional differential equations, Appl. Math. 24 (1996), 1-15.

[19] Bainov, D. D., Kolev, D. A. and Nakagawa, K., The control of the blowing-up time for the solution of of the semilinear parabolic equation with impulsive effect, J. Korean Math. Soc. 37 (2000), 793-802.

[20] Bainov, D. D., Minchev, E. and Nakagawa, K., Asymptotic behaviour of solutions of impulsive semilinear parabolic equations, Non. Anal. Theo. Meth. Appl. 30 (1997), 2725-2734.

[21] Bainov, D., Kamont, Z. and Minchev, E., On first order impulsive partial differential inequalities, Appl. Math. Comp. 61 (1994), 207-230.

[22] Barnett, S., Introduction to Mathematical Control Theory, Oxford Univ. Press, 1975.

[23] Benzaid, Z. and Sznaier, M., Constrained Controllability of Linear Impulse Differential Systems, IEEE Trans. Automat. Control 39 (1994), no. 5, 1064-1066.

[24] Bernfeld, S. R. and Lakshmikantham, V., An Introduction to Nonlinear Boundary Value Problems, Academic Press, New York and London, 1974.

[25] Borysenko S. D., Ciarletta M. and Lovane G., Integro-sum inequalities and motion stability of systems with impulse perturbations, Nonlinear Analysis 62 (2005),417-428.

[26] Borysenko, S. and Iovane, G., About some new integral inequalities of Wendroof type for discontinuous functions, Non. Anal. 66 (2007), 2190-2203.

[27] Boucherif, A. and Al-Malki, N., Nonlinear second order periodic boundary value problems, in Proc. Dynamical Systems and Appl. (Ed. H. Akca, A. Boucherif and V. Covachev), G. B. S. Publ., India, 2004, 178-188.

[28] Boucherif, A. and Fiagbedzi, Y. A., An integro-differential inequality with application, Tamui Oxford J. Math. Scie. 21 (2005), 217-224.

[29] Boucherif A. and Henderson, J., Topological methods in nonlinear boundary value problems, Nonlinear Times and Digest 1 (1994), 149-168.

[30] Boucherif, A., Impulsive Differential Equations. Conference Internationale surles Modeles, Methodes Mathematiques er Informatiotiques en Dynamique des Populations, 10-12 May, 2003, Tlemcen, Algeria.

[31] Boucherif, A., On the Topological Transversality Theorem of Granas and Some Applications, Annales de Mathematiques 6 (1999), 1-8.

[32] Bressan, A. and Rampazzo, F., Impulsive Control Systems with Commutative Vector Fields, J. Optim. Theory Appl., 71, No. 1 (1991), 67-83.

[33] Bressan, A. and Rampazzo, F., Impulsive Control Systems without Commutativity Assumptions, J. Optim. Theory Appl., 81, No. 3 (1994), 435-457.

[34] Bressan, A. and Rampazzo, F., On Differential Systems with Vector-valued Impulsive Controls, Boll. Un. Mat. Ital., 7, 2-B (1988), 641-656.

[35] Bressan, A., On Differential Systems with Impulsive Controls, Rend. Sem. Math. Univ. Padova, 78 (1987), 227-236.

[36] Byszewski, L., Impulsive nonlocal nonlinear parabolic differential problems, J. Appl. Math. Stochastic Anal., 6 (1993), 247-260.

[37] Camilli, F. and Falcone, M., Analysis and Approximation for The Impulse Control Problems on The Infinite Time Interval, Automat. Remot Control, 58, No. 7 (1997).

[38] Chan, C. Y. and Deng, K., Impulsive effects on global existence of solutions of semilinear heat equations, Nonl.Anal. 26 (1996), 1481-1489.

[39] Chen Dilan and Sun Jitao, On the conjunction practical stability and controllability of larg-scale impulsive control systems, J. Control Theory Appl., 2 (2005), 181-185.

[40] Chen, L. and Sun, J., Nonlinear boundary value problem of first order impulsive integro-differential equations, J. Comput. Appl. Math. 202 (2007), 392-401.

[41] Daryin, A. N. and Kurzhanski, A. B., Closed-loop Impulse Control of Oscillating Systems, 3rd IFAC Workshop on Periodic Control Systems, Vol. 3, 2007.

[42] Das, P. C. and Sharma, R. R., Existence and stability of measure differential equations, Czech. Math. J. 22 (1972), 145-158.

[43] Deng, L., Tan, Y. and Yu, Y., Oscilation criteria of solutions for a class of impulsive parabolic differential equation, Indian J. Pure Appl. Math. 33 (2002), 1147-1153.

[44] Elsgolts, L., Differential Equations and Calculus of Variations, MIR Publishers, Moscow, 1970.

[45] Erbe, L. H. and Liu, Xinzhi, Existence results for boundary value problems of second order impulsive differential equations, J. Math. Anal. Appl. 149(1990), 56-69.

[46] Erbe, L. H., Freedman, H. I., Liu, X. Z. and Wu, J. H., Comparison principles for impulsive parabolic equations with applications to models of single species growth, J. Austral. Math. Soc. Ser. B 32 (1991), 382-400.

[47] Filippova, T. F., State Estimation Problem for Impulsive Control Systems, Proceedings of the 10th Mediterranean Conference on Control and Automation - MED2002 Lisbon, Portugal, July 9-12, 2002.

[48] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[49] Fu, X. , Oscillation criteria for impulsive parabolic differential equations with delay, J. Math. Anal. Appl. 268 (2002), 647–664.

[50] Gallo A. and Piccirillo A. M., About new analogies of Gronwall-Bellman-Bihari type inequalities for discontinuous functions and estimated solutions for impulsive differential systems, Nonlinear Analysis 67 (2007), 15150-1559.

[51] George R. K., Nandakumaran A. K. and Arapostathis A., A Note on Controllability of Impulsive Systems, J. Math. Anal. Appl. 241 (2000), 276-283.

[52] Getz, W. M. and Martin, D. H., Optimal Control Systems with State Variable Jump Discontinuities, J. Optim. Theory Appl., 31, No. 2 (1980), 195-205.

[53] Granas, A., Homotopy Extension Theorem in Banach Spaces and Some of its Applications to the Theory of Nonlinear Equations, Bull. Acad. Pol. Sci. 7 (1959), 387-394.

[54] Groh, J., On The Optimal Control for Nonlinear Equations with Impulses. Math. Nach., 104 (1981), 41-48.

[55] Guan Z., Qian T. and Yu X., On Controllability and Observability for a Class of Impulsive Systems, Systems & control Lett. 47 (2002), 247-257.

[56] Guan, Z., Qian, T. and Yu, X., Controllability and Observability of Linear Time-Varying Impulsive Systems, IEEE Trans. Circut. Systems-I: Fundam. Theo. Appl. 49, No. 8 (2002),1198-1208.

[57] Guo, D., Existence of solutions of boundary value problems for nonlinear second order impulsive differential equation in Banach spaces, J. Math. Anal. Appl. 181(1994), 407-421.

[58] Guo, D., Second order impulsive integro-differential equations on unbounded domains in Banach spaces, Non. Anal. 35 (1999), 413-423.

[59] Han, X. and Lu, J., Impulsive Control Induced Effects on Dynamics of Single and Coupled ODE Systems, Nonlinear Dyn. 2009.

[60] He, L. and Liu, A.,Existence and uniqueness of solutions for nonlinear impulsive partial differential equations with delay, Non. Anal.: Real World Appl. 11 (2010), 952-958.

[61] He, Z. and He, X., Monotone iterative technique for impulsive integro-differential equations with periodic boundary conditions, Comp. Math. Appl. 48 (2004), 73-84.

[62] Hinpang, S., Xiang, X. and Sattayatham, P., Impulsive Periodic Control System with Parameter Perturbation, Thai J. Math. 4, No. 1 (2006), 107-125.

[63] Jankowski, T., Existence of solution for second order impulsive differential equations with deviating arguments, Non. Anal. 67 (2007), 1764-1774.

[64] Jitao Sun, L. C., Nonlinear boundary problem of first order impulsive integro-differential equations, J. Comp. Appl. Math. 202 (2007), 392-401.

[65] Kaufmanna, E. R., Kosmatov, N. and Raffoul, Y. N., A second-order boundary value problem with impulsive effects on an unbounded domain, Non. Anal. 69 (2008), 2924–2929.

[66] Khadra, A., Liu, X. and Shen, X., Application of Impulsive Synchronization to Communication Security, IEEE Transa. Circutis Systems I: Funda. Theo. Appl. 50, No. 3 (2003), 341-351.

[67] Kirane, M. and Rogovchenko, V., Comparison results for systems of impulse parabolic equations with applications to population dynamics, Non. Anal. Theo. Meth. Appl. 28 (1997), 263-276.

[68] Kolmogorov, A. N. and Fomin, S. V., Introductory Real Analysis, Prentice-Hall, Int., Eglewood Cliffs, Ney York, 1970.

[69] Kovalev A. M., Kravchenko N. V. and Nespirnyi V. N., Problems of Control and Stabilization of Impulsive-Control Dynamic systems with Application to Nonholonomic Mecanics, Automation and Remote Control 68, No. 8 (2007), 1444-1458.

[70] Ladyzhenskaya, O. A., Solonnikov, V. A. and Uraltseva, N. N., *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 [English translation: Amer. Math. Soc., Providence, R.I. 1968.]

[71] Lakmeche, A. and Boucherif, A., Boundary value problems for impulsive second order differential equations, Dynam. Cont. Discr. Impul. Syst. Series A: Math. Anal. 9 (2002), 313-319.

[72] Lakshmikantham, V., Bainov, D. D. and Simeonov, P. S., Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[73] Lee Y. and Liu Xi., Study of singular boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl. 331(2007), 159-176.

[74] Leela, S., Mcrae, F. A. and Sivasundaram, S., Controllability of Impulsive Differential Equations, J. Math. Analy. Appl. 177(1993), 24-30.

[75] Leela, S., Stability of measure differential equations, Pacific J. Math. 55 (1974), 489-498.

[76] Li, Z. G., Wen, C. Y. and Soh, Y. C., Analysis and Desing of Impulsive Control Systems, IEEE Trans. Automat. Control 46, No. 6 (2001), 894-897.

[77] Liu, X. Z., Impulsive Control and Optimization, Appl. Math. Compu., 73, No. 1 (1995), 77-98.

[78] Liu, X. Z., Practical Stabilization of Control Systems with Impulse Effects, J. Math. Anal. Appl., 166, No. 2 (1992), 563-576.

[79] Liz, E., Abstract monotone iterative techniques and applications to impulsive differential equations, Dayn. Cont. Disc. Impu. Sys. 3 (1997), 443-452.

[80] Manfenng H. and Zhenyuan X. , Analysis and desing of impulsive control systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 13A (2006), Part 2, suppl., 571-578.

[81] Mcrae, F. A., Practical stability of impulsive control systems, J. Math. Anal. Appl. 181 (1994), no. 3, 656-672.

[82] Medina, E. A., Linear Impulsive Control Systems: A Geometric Approach, Ph. D. Dissertation 2007.

[83] Medved, M., Singular integral inequalities with nonlinearities and integral equations with singular kernels, Nonlinear Oscillations 11 (2008), 70- 79.

[84] Miller, B. M. , The Generalized Solutions of Ordinary Differential Equations in the Impulse Control Problems, J. Math. Syst., Est. Control, 4, No. 3 (1994), 1-21.

[85] Miller, B. M. and Rubinovich, E. Ya., Impulsive control in continuous and discrete-continuous systems, Plenum Publisher, New York, 2003.

[86] Miller, B. M., Method of Discontinuous Time Change in Problems of Control for Impulse and Discrete-Continuous Systems, Automat. Remot Control, 54 (1993), 1727-1750.

[87] Mil'man, V. D. and Myshkis, A. D., On the stability of motion in the presence of impulses (in Russian), Sibirsk. Mat. Z. 1 (1960), 233-137.

[88] Mitropolskiy Yu. A., Iovane G. and Borysenko S. D., About a generalization of Bellman-Bihari type inequalities for discontinuous functions and their applications, Nonlinear Analysis 66 (2007), 2140-2165.

[89] Motta, M. and Rampazzo, F., Daynamic Programming for Nonlinear Systems Driven By Ordinary and Impulsive Control, SIAM J. Control Optim., 34, No. 1 (1996), 199-225.

[90] Murray, J. M., Existence Theorems for Optimal Control and Calculus of Variations Problems Where the State Can Jump, SIAM J. Control Optim., 24, No. 3 (1986), 412-438.

[91] Myshkis, A. D., On a recurrently defined sequence, J. Difference Equ. Appl. 3 (1997), 89-91.

[92] Nakagawa, K., Asymptotic behaviour of solutions of an impulsive semilinear parabolic cauchy problem, 8th Int. Coll. on Diff. Equ. (1998), 335-340.

[93] Natanson, I. G., Theory of Functions of a Real Variable, Frederik Ungar, New York, (1955, 1960), Vols. 1 and 2.

[94] Nieto, J. J., Basic theory for nonresonance impulsive periodic problems of first order, J. Math. Anal. Appl. 205 (1997), 423-433.

[95] Pandit, S. G. and Deo, S. G., Differential Systems Involving Impulses, Lect. Notes Math. 954, Springer Verlag, Berlin-New York, 1982.

[96] Pandit, S. G., Differential systems with impulsive perturbations, Pacific J. Math., 86 (1980), 553-560.

[97] Pandit, S. G., On the stability of impulsively perturbed differential systems, Bull. Austral. Math. Soc. 17 (1977), 423-432.

[98] Pao, C. V., Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[99] Peng, Y., Xiang, X., and Wei, W., Necessary conditions of optimality for second-order nonlinear impulsive differential equations, Hindawi Publishing Corporation Adva. Diff. Equ. 2007, Article ID 40160, 17 pages.

[100] Pereira F. L. and Silvo G. N., Stability for impulsive control systems, Dyn. Syst. 17, No. 4 (2002), 421- 434.

[101] Rachunkova I. and Tomecek J., Singular Dirichlet problem for ordinary differential equation with impulses, Nonlinear Analysis 65 (2006), 210-229.

[102] Rao, V. S. H., Note on asymptotic equivalence for measure differential equations, Bull. Austral. Math. Soc. 18 (1978), 455-460.

[103] Rao, V. S. H., On boundedness of impulsively perturbed systems, Bull. Austral. Math. Soc. 18 (1978), 237-242.

[104] Rogovchenko, Y. V., Comments on " Well-posedness of impulsive problems for nonlinear parabolic equations", Nonlinear Studies, 10 (2003), 273-277.

[105] Rogovchenko, Y. V., Comparison princibles for systems of impulsive parabolic equations, Annali Math. Pur. Appl., Vol. CLXX (1996), 311-328.

[106] Rudin, W., Real and Complex Analysis, McGraw-Hill, New York (1966).

[107] Samoilenko, A. M. and Perestyuk, N. A., impulsive differential equations, World Scientific, Singapore, 1995.

[108] Schwabik, Š, Generalized Ordinary Differential Equations, World Scintific, London, 1992.

[109] Shen, J., Luo, Z. and Liu, X., Impulsive stabilization of functional differential equations via Liapunov functionals, J. Math. Anal. Appl. 240 (1999), 1-5.

[110] Shilov, G. E. and Gurevich, B. L., Integral, Measure and Derivative: A Unified Approach, Prentice-Hall, Inc., Englewood Cliffs, Ney York, 1966.

[111] Shin C. E. and Ryu J. H., Generalized solution of time dependent impulsive control system corresponding to vector-valued controls of bounded variation, Bull. Korean. Math. Soc. 37 (2000), no. 2, 229-247.

[112] Shin, C. E. and Hyun, R. J. , Generalized solution of time dependent impulsive control system, Rend. Sem. Mat. Univ. Padova 101 (1999),161-178.

[113] Shin, C. E., Generalized solution of impulsive control system corresponding to controls of bounded variation, J. Korean. Math. Soc. 34, No. 3 (1997), 581-598.

[114] Skóra, L., Theorems on impulsive parabolic differential-functional inequalities, Demonstratio Mathematica, Vol. XXXII (1999), 55-66.

[115] Sun J., Zhang Y. and Wu Q., Less Conservative Conditions for Asymptotic Stability of Impulsive Control Systems, IEEE Trans. Automat. Control 48, No. 5(2003), 829-831.

[116] Sun, S. and Chen, L., Dynamic Behaviors of Monod Type Chemostat Model with Impulsive Perturbation on the Nutrient Concentration, J. Math. Chemistry 42, No. 4 (2007), 837-847.

[117] Tamura, T., Maximizing the Growth Rate of a Portfolio with Fixed and Proportional Transaction Costs, Appl. Math. Optim. 54 (2006), 95–116.

[118] Tatar, N., An impulsive nonlinear singular version of the Gronwall-Bihari inequality, J. Ineq. Appl. 2006, Article ID 84561, 1-12.

[119] Taylor, A. E., General Theory of Functions and Integration, Blaisdell Publishing Company, 1965.

[120] Tomeček, J., Nonlinear boundary value problem for nonlinear second order differential equations with impulses, Electronic J. Qual. Theo. Diff. Equ. 10 (2005), 1-22.

[121] Vlasenko, L. A., Myshkis, A. D. and Rutkas, A. G., On a class of differential equations of parabolic type with impulsive action, Diff. Equa. 44 (2008), 231-240.

[122] Xia Z. and Sun J., On Practical Stability for Larg Scale Impulsive Control System in Terms of Two Measurements.Proceedings of the 5th Worled Congress on Intelligent Control and Automation, 2004, China, 142- 145.

[123] Xia Z. and Sun J., Practical stability analysis of impulsive control system, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 2004, Added Volume, 101-107. Proceedings of the 5th Worled Congress on Intelligent Control and Automation, 2004, China, 142- 145.

[124] Xin-ming, C. Zhi-hong, G. and Xin-zhi, L., Decentralized impulsive control for a class of uncertain interconnected systems, J. Zhejiang Univ. SCI. 5, No. 3 (2004), 274-282.

[125] Yang, T. and Chua, L. O., Impulsive Stabilization for Control and Synchronization of Chaotic Systems: Theory and Application to Secure Communication, IEEE Trans. Circut. Systems-I: Fundam. Theo. Appl. 44 (1997), no. 10, 976-988.

[126] Yang, T., Impulsive control theory, Springer, Berlin, 2001.

[127] Yang, T., Impulsive control, Springer, IEEE Trans. Automat. Control 44 (1999), no. 5, 1081-1083.

[128] Yang, X. and Shen, J., Periodic boundary value problems for second-order impulsive integro-differential equations, J. Compu. Appl. Math. 209 (2007), 176 –186.

[129] Young, E. C., Partial Differential Equations, An Introduction. Allyn and Bacin, Inc., Boston, 1972.

[130] Zavalishchin, S. T., Impulse Dynamic Systems and Applications to Mathematical Economics, Dynamic Systems and Applications, 3, No. 3 (1994), 443-450.

[131] Zhang, H., Chen, L. and Georgescu, P., Impulsive Control Strategies for Pest Management, J. Biolog. Syst. 15, No. 2 (2007), 235-260.

[132] Zhang, Y., Analysis and Design of Set-Stability for Impulsive Control Systems, Phys. Lett. A 370 (2007), 459–464.

[133] Zhang, Y. and Sun, J., Stability of Impulsive Infinite Delay Differential Equations, Appl. Math. Lett. 19 (2006), 1100-1106.

[134] Zhang, W., Agarwal, R. P. and Akin-Bohner, E., On well-posedness of impulsive problems for nonlinear parabolic equations, Nonlinear Studies 9 (2002), 145-153.

[135] Zhao H. and Feng E., Stability of impulsive control systems with variable times, J. Appl. Math. Comput. 23(2007), no. 1-2, 345-352.

Vita

* Ali Saeed Salem Al-Talhan Al-Qahtani.

* Born in the south of Saudi Arabia, in 1972.

* Received B. Sc. degree in mathematics from King Saud University in 1994.

* Taught Mathematics at intermediate and secondary schools.

* Joined Mathematical Science Department in the Teachers College in Abha as a Graduate Assistant in 1998.

* Received M. Sc. Degree in Mathematics in May 2004.

* Worked as a Lecturer in the Department of Mathematics, Abha Teachers College, Abha, 2005 - 2008.

* Joined as Lecturer the Department of Mathematics, King Khaled University, Abha, Saudi Arabia, 2008.

* Email: alitalhan@hotmail.com, algahtani@kku.edu.sa.